

The near-critical planar FK-Ising model

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Abstract

We study the near-critical FK-Ising model. First, a determination of the correlation length defined via crossing probabilities is provided. Second, a striking phenomenon about the near-critical behavior of FK-Ising is highlighted, which is completely missing from the case of standard percolation: in any monotone coupling of FK configurations ω_p (*e.g.*, in the one introduced in [Gri95]), as one raises p near p_c , the new edges arrive in a fascinating self-organized way, so that the correlation length is not governed anymore by the number of pivotal edges at criticality. In particular, it is smaller than the heat-bath dynamical correlation length determined in the forthcoming [GP].

We also include a discussion of near-critical and dynamical regimes for general random-cluster models. For the heat-bath dynamics in critical random-cluster models, we conjecture that there is a regime of values of cluster-weights q where there exist macroscopic pivots yet there are no exceptional times. These are the first natural models that are expected to be noise sensitive but not dynamically sensitive.

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1 Introduction

Phase transition in the random cluster model. The random-cluster model with parameters $p \in [0, 1]$ and $q \geq 1$ ¹ is a probability measure on subgraphs of a finite graph $G = (V, E)$, defined for all $\omega \subset E$ by

$$\phi_{p,q}(\omega) := \frac{p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}}q^{\#\text{ clusters}}}{Z_{p,q}},$$

where $Z_{p,q}$ is the normalization constant such that $\phi_{p,q}$ is a probability measure. The most classical example of the random-cluster model is bond percolation, which corresponds to the $q = 1$ case. Infinite volume measures can be constructed using limits of the above measures along exhaustions by finite subsets (with different boundary conditions: free, wired, etc). Random-cluster models exhibit a phase transition at some critical parameter $p_c = p_c(q)$. On \mathbb{Z}^d , this value does not depend on which infinite volume limit we are using, and, as in standard percolation, below this threshold, clusters are almost surely finite, while above this threshold, there exists (a.s.) a unique infinite cluster. See Subsection 2.1 for details and references.

The critical parameter is known to be equal to $1/2$ for bond percolation on the square lattice. For the random-cluster model with parameter $q = 2$ (also called *FK-Ising*), $p_c(2) = \frac{\sqrt{2}}{1+\sqrt{2}}$ is known since Onsager [Ons44] (it is connected via the Edwards-Sokal coupling to the critical temperature of the Ising model). See also the recent [BDC10b] for an alternative proof of this fact. More recently, the general equality $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$ was proved for every $q \geq 1$ in [BDC10a].

Mathematicians and physicists are interested in the properties of the critical phase $(p, q) = (p_c(q), q)$ itself. Very little is known for general values of q , and only $q = 1$ and $q = 2$ are understood in a satisfactory fashion. For these two cases, the phase transition is known to be of second order (*i.e.* continuous): at p_c , almost surely there are only finite clusters. For $q = 2$, Smirnov proved the **conformal invariance** of certain macroscopic observables [Smi10, Smi], which, combined with the Schramm-Löwner Evolution [Sch00], can be used to construct **continuum scaling limits** that retain the macroscopic cluster structure. For $q = 1$, site percolation on the triangular lattice (a related model) has been proved to be conformally invariant as well [Smi01].

Beyond the understanding of the critical phase, the principal goal of statistical physics is to study the phase transition itself, and in particular the behavior of macroscopic properties (for instance, the density of the infinite-cluster for $p > p_c(q)$). It is possible to relate the critical regime to these thermodynamical properties via the study of the so-called *near-critical regime*. This regime was investigated in [Kes87] in the case of percolation. Many works followed afterward, culminating in a rather good understanding of dynamical and near-critical phenomena in standard percolation [GPS10a, GPS10b, GPS]. The goal of this article is to discuss the near-critical regime in the random-cluster case, and more precisely in the FK-Ising case.

Near-critical regime and correlation length. The near-critical regime is the study of the random-cluster model of edge-parameter p in the box of size L when (p, L) goes to (p_c, ∞) . Note that, on the one hand, if p goes to p_c very quickly the configuration in the

¹In general one could take $q > 0$, but we will assume $q \geq 1$ here, in order for the **FKG inequality** to hold, see [Gri06].

box of size L will look critical. On the other hand, if p goes to p_c (from above) too slowly, the random-cluster model will look supercritical. The typical scale $L = L(p)$ separating these two regimes is called the **correlation length** (or **characteristic length**). In rough terms, if p is slightly above $p_c(2) = \sqrt{2}/(1 + \sqrt{2})$, the correlation length $L(p)$ is the scale below which things still look somewhat *critical* and above which the infinite cluster starts being *visible*. In the subcritical regime, it corresponds to the scale above which the fact that p is subcritical becomes apparent.

Definition of correlation length in the case of percolation ($q = 1$). The critical regime is often characterized by the fact that crossing probabilities remain strictly between 0 and 1. Formally, consider *rectangles* R of the form $[0, n] \times [0, m]$ for $n, m > 0$, and translations of them. We denote by $\mathcal{C}_v(R)$ the event that there exists a *vertical crossing* in R , a path from the bottom side $[0, n] \times \{0\}$ to the top side $[0, n] \times \{m\}$ that consists only of open edges. The celebrated Russo-Seymour-Welsh theorem shows that in the case of critical percolation, crossing probabilities of rectangles of bounded aspect ratio remain bounded away from 0 and 1. A natural way of describing the picture as being critical is to check that crossing probabilities are neither near 0 nor near 1. Mathematically, it is thus natural to define the correlation length for every $p < p_c = 1/2$ and $\varepsilon > 0$ as

$$L_\varepsilon(p) := \inf \{n > 0 : \mathbb{P}_p(\mathcal{C}_v([0, n]^2)) \leq \varepsilon\},$$

and, when $p > p_c = 1/2$, as $L_\varepsilon(p) := L_\varepsilon(1 - p)$, where $1 - p$ is the dual edge-weight. The dependence on ε is not relevant, since it can be proved ([Kes87, Nol08]) that for any $\varepsilon > 0$,

$$L_{1/4}(p) \asymp L_\varepsilon(p),$$

where \asymp means that there exist constants $0 < A_\varepsilon, B_\varepsilon < \infty$ such that

$$A_\varepsilon L_{1/4}(p) \leq L_\varepsilon(p) \leq B_\varepsilon L_{1/4}(p).$$

The correlation length was shown to behave like $|p - p_c(1)|^{-4/3+o(1)}$ in the case of percolation [SW01].

Correlation length for FK-Ising percolation ($q = 2$). The first result of this paper is a determination of the behavior of $L(p)$ when p goes to p_c for $q = 2$. Before stating the main result, let us give a proper definition of *correlation length* in this setting. Since the Russo-Seymour-Welsh theorem has been generalized to the FK-Ising case in [DCHN10] (see Theorem 2.1 in the present text), it is natural to characterize the *critical regime* once again by the fact that crossing probabilities remain strictly between 0 and 1. An important difference from the $q = 1$ case is that one has to take into account the effect of *boundary conditions* (see Subsection 2.1 for precise definitions):

Definition 1.1 (Correlation length). *Fix $q = 2$ and $\rho > 0$. For any $n \geq 1$, let R_n be the rectangle $[0, n] \times [0, \rho n]$.*

If $p < p_c(2)$, for every $\varepsilon > 0$ and boundary condition ξ , define

$$L_{\rho, \varepsilon}^\xi(p) := \inf \left\{ n > 0 : \phi_{p, 2, R_n}^\xi(\mathcal{C}_v(R_n)) \leq \varepsilon \right\},$$

where $\phi_{R, p, 2}^\xi$ denotes the random-cluster measure on R with parameters $(p, 2)$ and boundary condition ξ . If $p > p_c(2)$, define similarly

$$L_{\rho, \varepsilon}^\xi(p) := \inf \left\{ n > 0 : \phi_{p, 2, R_n}^\xi(\mathcal{C}_v(R_n)) \geq 1 - \varepsilon \right\}.$$

Our main result on the correlation length can be stated as follows.

Theorem 1.2. *Fix $q = 2$. For every $\varepsilon, \rho > 0$, there is a constant $c = c(\varepsilon, \rho)$ such that*

$$c \frac{1}{|p - p_c|} \leq L_{\rho, \varepsilon}^\xi(p) \leq c^{-1} \frac{1}{|p - p_c|} \sqrt{\log \frac{1}{|p - p_c|}}$$

for all $p \neq p_c$, whatever the choice of the boundary condition ξ is.

Note that the left-hand side of the previous theorem has the following reformulation, which we state as a theorem of its own (this result is interesting on its own since it provides estimates on crossing probabilities which are uniform in boundary conditions away from the critical point):

Theorem 1.3 (RSW-type crossing bounds). *For $\lambda > 0$ and $\rho \geq 1$, there exist two constants $0 < c_- \leq c_+ < 1$ such that for any rectangle R with side lengths n and $m \in [\frac{1}{\rho}n, \rho n]$, any $p \in [p_c - \frac{\lambda}{n}, p_c + \frac{\lambda}{n}]$ and any boundary condition ξ , one has*

$$c_- \leq \phi_{R, p, 2}^\xi(\mathcal{C}_v(R)) \leq c_+.$$

The main ingredient of the proof of the latter theorem (and the most interesting one) is Smirnov's fermionic observable. This observable is defined in Dobrushin domains (with a free and a wired boundary arc), and is a key ingredient in the proof of conformal invariance at criticality. Nevertheless, its importance goes far beyond that proof, in particular because it can be related to connectivity properties of the FK-Ising model. We study its properties away from the critical point (developing further the methods of [BDC10b]), and estimate its behavior near the free arc of Dobrushin domains. It implies estimates on the probability for sites of the free arc to be connected to the wired arc. This, as in [DCHN10], allows us to perform a second-moment estimate on the number of connections between sites of the free arc and the wired arc, therefore implying crossing probabilities in Dobrushin domains. All that remains is to get rid of the Dobrushin boundary conditions (which is not as simple as one might hope; in particular, harder than in [DCHN10]), in order to obtain crossing probabilities with free boundary conditions.

Using conformal invariance techniques, Chelkak, Hongler and Izyurov has recently proved that

$$\phi_{p_c, q=2}(0 \leftrightarrow \partial[-n, n]^2) \sim C n^{-1/8}, \quad (1.1)$$

together with the appropriate (conformally invariant) version for general domains [CHI12]. Together with Theorem 1.2, this implies:

Theorem 1.4. *Assuming (1.1), there exists a constant $c > 0$ such that if $p > p_c(2)$,*

$$\phi_{p, 2}(0 \leftrightarrow \infty) \geq c \left(\frac{|p - p_c|}{\sqrt{\log 1/|p - p_c|}} \right)^{1/8}.$$

The critical 1-arm exponent (1.1) and the off-critical result

$$\langle \sigma_0 \rangle_\beta^+ = \phi_{p, 2}(0 \leftrightarrow \infty) \asymp |\beta - \beta_c|^{1/8} \quad (\text{as } \beta \searrow \beta_c)$$

go back to Onsager [Ons44]. These two results more-or-less imply the correlation length

$$L(p) \asymp \frac{1}{|p - p_c|},$$

with more precise and direct arguments presented in [FF69] and [Kad69] (see also [BDC10b]). However, their notion of correlation length is different and more restricted than the one in our Theorem 1.2, and its equivalence with our notion is not known. Furthermore, our proof of Theorem 1.4 is also of some value, since the result of [CHI12] and the techniques in this paper extend to isoradial graphs (with additional work) while Onsager’s technology is restricted to the square lattice.

The random-cluster model through its phase transition. The previous way to look at the near-critical regime may seem slightly artificial. It is more natural to study the random-cluster model through its phase transition by constructing a monotone coupling of random-cluster models with fixed cluster-weight $q \geq 1$. Then, properties of the monotone coupling (which can be thought of as a dynamics following the evolution of p between 0 and 1) near p_c will describe the near-critical regime.

In the case of standard bond percolation ($q = 1$), such a monotone coupling simply consists of i.i.d. Uniform $[0, 1]$ labels on the edges, and a percolation configuration ω_p of density p is the set of bonds with labels at most p . It is straightforward to interpret this coupling as an asymmetric dynamical percolation: starting from critical percolation at time zero, as time goes on, whenever the clock of a bond rings, we open that bond; we can also run time backwards and close the bonds that ring. Now, the question is: in this monotone coupling, how fast does the system enter the supercritical and subcritical regimes as p changes near p_c ?

This **near-critical window** in percolation was studied by Kesten in [Kes87], then by [BCKS01, Nol08, NW09, GPS10b, GPS]. It turns out that its size is governed by the expected number of **macroscopically pivotal edges** at criticality, *i.e.*, edges having four alternating (between dual and primal) open paths starting there and going to a macroscopic distance. Let us describe briefly how this mechanism works, since one of our main goals in this article is to point out that, in the case of FK-percolation ($q > 1$), a striking new phenomenon appears.

Let $\alpha_4(n)$ be the probability at criticality that an edge has four alternating paths going to distance n . Getting from ω_{p_c} to $\omega_{p_c+\Delta p}$ in the box of size L , the system is moving out of stationarity, and roughly $L^2\Delta p$ edges are switched from closed to open (we are assuming $\Delta p > 0$). The expected number of opened edges that were closed macroscopic pivots in the *initial* configuration (preventing macroscopic open paths) is about $L^2\alpha_4(L)\Delta p$. If $L^2\alpha_4(L)\Delta p \gg 1$, then many of these initial macroscopic pivots have become open with good probability. It is not very hard to show that this means that the window of size L has become well-connected, hence we have left the near-critical regime.

What can we say if $L^2\alpha_4(L)\Delta p \ll 1$? The number of initial macroscopic pivots that have switched is small, but maybe many new pivots have appeared during the dynamics, which could have switched then, establishing macroscopic open connections. To formulate the same issue from a different point of view, if the dynamics, instead of switching always from closed to open, was symmetric dynamical percolation, then the system would be critical all the time, and, using Fubini and the linearity of expectation, the expected number of macroscopic pivotal switches (*i.e.*, flips of edges that are macroscopically pivotal *at the moment of the flip*) in time Δp would be $L^2\alpha_4(L)\Delta p$. If this expectation is small, then the probability of having any macroscopic pivotal switches is also small, hence the system indeed has not changed macroscopically. However, the asymmetric near-critical dynamics is slowly moving out of criticality, which could have an effect on the number of pivots, speeding up changes.

Nevertheless, Kesten proved the following **near-critical stability** result: as long as $L^2\alpha_4(L)\Delta p = O(1)$, there are not many more pivotal points in ω_p than at criticality, hence, despite the monotonicity of the dynamics, changes do not speed up significantly compared to symmetric dynamical percolation, and hence the macroscopic geometry starts changing significantly only at $L^2\alpha_4(L)\Delta p \asymp 1$. The proof in [Kes87] employs Russo's influence formula and differential inequalities, discussed more in Subsection 3.1. There is a related but more geometric approach in [GPS], which provides some additional insight into the mechanism governing the near-critical regime that will be crucial in understanding the case of FK-percolation.

Namely, [GPS] proves the following **dynamical stability** result about symmetric dynamical percolation (needed to establish that dynamical percolation has a scaling limit that is a Markov process describing the changes of the macroscopic connectivity structure): as long as $L^2\alpha_4(L)\Delta t = O(1)$, in order to describe the macroscopic structure of $\omega_{\Delta t}$, it is enough to know the macroscopic structure of ω_0 and to follow the flips experienced by all initial macroscopic pivotals. In other words, there are no cascades of information from small to much larger scales, i.e., edges initially pivotal only on a “mesoscopic” scale are unlikely to have a macroscopic impact in the dynamics within the given time frame. This is proved using induction, with a careful summation over all possible ways in which “smaller” pivotals can make a big difference. For this summation argument, it is of course essential (and this will be important later) that edges are switching independently from each other. Now, the same argument applies to the asymmetric dynamics, and gives the near-critical stability that we stated above: in order to change the macroscopic connectivity structure, initial macroscopic pivotals need to be flipped.

Summarizing, the scale at which the critical regime becomes the supercritical regime is given by $L^2\alpha_4(L)\Delta p \asymp 1$. The same reasoning can be applied to the subcritical regime. In particular, the correlation length is given (up to constants) by Kesten's relation

$$(L_\varepsilon(p))^2\alpha_4(L_\varepsilon(p))|p - p_c| \asymp 1, \quad (1.2)$$

where the constant factors of \asymp depend only on the value of ε .

Going back to dynamical percolation for a second, it is not very hard to show that $L^2\alpha_4(L)\Delta t \geq c > 0$ implies that $\omega_{\Delta t}$ is already somewhat different from ω_0 , while the much harder [GPS10a] says that $L^2\alpha_4(L)\Delta t \gg 1$ implies that the macroscopic structure of $\omega_{\Delta t}$ has completely forgotten the initial configuration ω_0 . But even without the latter result, we see that, for any time interval Δt , if $L = L(\Delta t)$ is given by $L^2\alpha_4(L)\Delta t \asymp 1$, then, below this scale L , the macroscopic structure is essentially untouched, while, at larger scales, changes are already visible. So, this L may be called the **dynamical correlation length**.

The main principle we shall extract from this discussion can be stated as follows:

Phenomenon 1.5. *In percolation ($q = 1$), due to near-critical stability, the near-critical behavior is governed by the number of pivotal points at criticality. The near-critical correlation length coincides with the dynamical correlation length.*

To our knowledge, it has been widely believed in the community that basically the same mechanism should hold in the case of random-cluster models. Namely, once we understand the geometry of the set of pivotal points at criticality, we may readily deduce information on the near-critical behavior. However, this is not the case.

Let us consider the case of the FK-Ising. It is shown in [DCG] that the critical FK-Ising probability $\alpha_4^{\text{FK}}(n)$ for a site to be pivotal behaves like $n^{-35/24+o(1)}$ when n goes to

infinity. If pivotal points were governing the near-critical regime, the correlation length should satisfy

$$(L_\varepsilon^{\text{FK}}(p))^2 \alpha_4(L_\varepsilon^{\text{FK}}(p)) |p - p_c| \asymp 1, \quad (1.3)$$

which would give

$$L_\varepsilon^{\text{FK}}(p) = |p - p_c(2)|^{-\frac{24}{13} + o(1)} \gg |p - p_c(2)|^{-1}, \quad (1.4)$$

contradicting Theorem 1.2.

What goes wrong with the percolation arguments? First, there is a basic phenomenon in the $\text{FK}(p, q)$ models for $q \geq 2$ that is very relevant to the above discussion: the difference between the average densities of edges for $p = p_c(q) + \Delta p$ and $p = p_c(q)$ is not proportional to Δp , but larger than that, with an exponent given by the so-called **specific heat** of the model. (We will discuss this in more detail in Subsection 3.2.) A first guess could be that the discrepancy in (1.4) is a result of the fact that Δp is not the density of the new edges arriving, and this should have been taken into account in the computation using the pivotal exponent. However, this is only partially right: the specific heat exponent itself is not large enough to account for this discrepancy (in fact, for $q = 2$ it equals 0 — there is only a logarithmic blow-up). In fact, a **self-organizational mechanism** kicks in. In standard percolation, on the way from ω_{p_c} to $\omega_{p_c + \Delta p}$ in the asymmetric dynamics, new edges arrive in a “Poissonian” way. This is no longer the case with FK-Ising: the arriving edges tend to prefer “strategic” locations, which are pivotal at large scales, thereby speeding up the dynamics compared to what could be guessed from the number of pivots at criticality. In other words, near p_c , the arriving edges depend in a very sensitive way on the current configuration. This subtle balance between the current configuration and the conditional law of the arriving edges is representative of a self-organized mechanism.

How do we know about this self-organization? Similarly to dynamical percolation, there is a natural dynamics with the random-cluster model $\text{FK}(p, q)$ as stationary distribution, called the heat-bath dynamics or Sweeny algorithm (see Definition 3.3): edges have independent exponential clocks, and when the clock of $e = \langle x, y \rangle$ rings, the state of e is updated according to the $\text{FK}(p, q)$ measure conditioned on the rest of the configuration ω , of which the only relevant information is whether x and y are connected in $\omega \setminus \{e\}$. Now, in [GP], the analogue of the above-mentioned dynamical stability result of [GPS] is proved for the critical FK-Ising model $\text{FK}(p_c(2), 2)$. Thus, if there was *any* monotone coupling of the near-critical $\text{FK}(p, 2)$ models in which new edges arrived in a Poissonian way similar to the heat bath dynamics, with clock rates and resampling probabilities bounded away from 0, then the same argument would apply, and near-critical stability would hold, proving (1.3). Since this is not the case, the Poissonian picture can be ruled out:

Phenomenon 1.6. *The correlation length in FK-Ising is much smaller than what the intuition coming from standard percolation ($q = 1$) would predict. In other words, as one raises the parameter p , the supercritical regime appears “faster” than what would be dictated simply by the number of pivotal edges at criticality: new edges arrive in a very non-uniform manner, and a striking **self-organized near-criticality** appears.*

The proof that this self-organization mechanism must play an important role in any monotone coupling depends on the forthcoming dynamical FK-paper [GP], but we nevertheless wanted to draw attention to this very surprising near-critical corollary already in this paper. A particular realization of this phenomenon that we can offer here is the following. In Subsection 3.3, we will introduce Grimmett’s monotone coupling of the $\text{FK}(p, q)$

configurations as p varies from 0 to 1 and $q \geq 1$ is fixed. In Subsection 3.4, we will present a concrete way in which self-organization works in this coupling. However, most of the self-organization scheme remains to be understood; some questions are presented in Subsection 3.5. Furthermore, it is interesting to note that there is a hyperscaling relation between specific heat and correlation length, hence the specific heat determines not only the number of new edges arriving, but also the correlation length (in some way that appears to be independent of the self-organization scheme). We will briefly discuss this hyperscaling in Subsection 4.3.

In Subsection 3.6, we present another point of view that might explain the discrepancy (1.4): for $q > 1$, Russo's formula used in Kesten's proof has to be modified. Namely, the influence of an edge on the event that a box of size n is crossed does not coincide with the probability for that edge to be pivotal, as it was the case for $q = 1$. Let us define the critical exponent $\iota(q)$ by assuming that the above influence of an edge behaves like $n^{-\iota(q)}$ at criticality. Kesten's scaling relation for $q = 1$ was $(2 - \xi_4(q))\nu(q) = 1$, where $\xi_4(q)$ and $\nu(q)$ are the critical exponents of the pivotal event and the correlation length, respectively, coming from (1.2). This will remain valid for $q > 1$ only if the critical exponent $\xi_4(q)$ is replaced by the exponent $\iota(q)$ governing the behavior of the influence. This subtlety and the fact that $\xi_4(q) \neq \iota(q)$ seem to be new.

On the dynamical and near-critical behavior for other values of q . Finally, we investigate what happens for other values of q . Since the mathematical understanding of critical FK(q) models is very limited when $q \notin \{0, 1, 2\}$, the study relies on predictions from physics. We will mostly focus on the case $q \in [1, 4]$, where the FKG inequality holds and the phase transition is conjectured to be continuous (*i.e.*, there is a unique infinite-volume measure at criticality, having no infinite cluster). It is therefore natural to consider the near-critical regime. In this case yet again, pivotal points do not seem to control the behavior of the near-critical regime. Critical exponents indeed violate Kesten's relations. See Section 4.

We will also discuss briefly noise-sensitivity and dynamical sensitivity of random-cluster models with $q \in [1, 4]$. Note that, as it is already apparent from the above discussion, the study of the near-critical regime of percolation (especially in [GPS10b, GPS]) was conducted in parallel to the study of dynamical percolation [SS10, GPS10a]. For random-cluster models with $q \in (1, 4]$, one may indeed study the influence of pivotal edges on the existence of exceptional times in the heat-bath dynamics: times for which an infinite cluster exists. As for the near-critical regime, the situation seems much more complicated than the percolation one, although for different reasons. (For dynamical sensitivity, instead of the true near-critical model, a “fake, Poissonian” near-critical model appears to be more relevant.) In particular, the existence of pivotal edges is not equivalent (conjecturally) to the dynamical sensitivity of the model (it is expected to be equivalent to noise sensitivity, though). We refer to Subsection 5.1 for further details on these phenomena.

Random-cluster models with $q > 4$ do not have the same behavior as those with $q \leq 4$. Indeed, the phase transition is conjectured to be of first order, and the near-critical regime exists only in a restricted sense: there is a non-trivial critical window in which a finite system becomes supercritical from subcritical, but the correlation length remains bounded for all p . In addition, the models are not expected to be noise-sensitive or dynamically sensitive. Subsection 5.2 is devoted to their study.

Organization of the paper. In Section 2, we begin with an exposition of basic properties of random-cluster models, and define Smirnov's so-called fermionic observable. We use it away from criticality to prove Theorem 1.3, then we deduce Theorems 1.2 and 1.4.

In Section 3, we study the self-organized phenomenon in detail. We start by explaining why pivotal points are crucial in the understanding of the near-critical regime of percolation. First, we discuss the role of specific heat, then present Grimmett's monotone coupling, under which the evolution of the random-cluster model through its phase transition can be monitored. Finally we explain how the self-organized phenomenon acts concretely.

Section 4 contains a discussion of other values of q , while Section 5 mentions the interesting case of dynamical random-cluster models.

2 Proofs of the main results on the correlation length (Theorems 1.3, 1.2 and 1.4)

In this section, a point will be identified with its complex coordinate.

2.1 Basic properties of random-cluster models

The random-cluster measure can be defined on any graph. However, we restrict ourselves to the standard square lattice \mathbb{Z}^2 . With a tiny abuse of notation, we will use $V(\mathbb{Z}^2)$ or just \mathbb{Z}^2 for the set of sites, and $E(\mathbb{Z}^2)$ for the set of bonds. In this paper, G will always denote a connected subgraph of \mathbb{Z}^2 , *i.e.*, a subset of vertices together with all the bonds between them. We denote by ∂G the (inner) boundary of G , *i.e.*, the set of sites of G linked by a bond to a site of $\mathbb{Z}^2 \setminus G$.

A *configuration* ω on G is a random subgraph of G , having the same sites and a subset of its bonds. We will call the bonds belonging to ω *open*, the others *closed*. Two sites a and b are said to be *connected* (denoted by $a \leftrightarrow b$), if there is an *open path* — a path composed of open bonds only — connecting them. The (maximal) connected components will be called *clusters*. More generally, we extend this definition and notation to sets in a straightforward way.

A *boundary condition* ξ is a partition of ∂G . We denote by $\omega \cup \xi$ the graph obtained from the configuration ω by identifying (or *wiring*) the vertices in ξ that belong to the same class of ξ . A boundary condition encodes the way in which sites are connected outside of G . Alternatively, one can see it as a collection of *abstract bonds* connecting the vertices in each of the classes to each other. We still denote by $\omega \cup \xi$ the graph obtained by adding the new bonds in ξ to the configuration ω , since this will not lead to confusion. Let $o(\omega)$ (resp. $c(\omega)$) denote the number of open (resp. closed) bonds of ω and $k(\omega, \xi)$ the number of connected components of $\omega \cup \xi$. The probability measure $\phi_{G,p,q}^\xi$ of the random-cluster model on a *finite* subgraph G with parameters $p \in [0, 1]$ and $q \in (0, \infty)$ and boundary conditions ξ is defined by

$$\phi_{G,p,q}^\xi(\{\omega\}) := \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k(\omega,\xi)}}{Z_{G,p,q}^\xi}, \quad (2.1)$$

for any subgraph ω of G , where $Z_{G,p,q}^\xi$ is a normalizing constant known as the *partition function*. When there is no possible confusion, we will drop the reference to parameters in the notation.

The domain Markov property. One can encode, using an appropriate boundary condition ξ , the influence of the configuration outside a sub-graph on the measure within it. Consider a graph $G = (V, E)$ and a random-cluster measure $\phi_{G,p,q}^\psi$ on it. For $F \subset E$, consider G' with F as the set of edges and the endpoints of it as the set of sites. Then, the restriction to G' of $\phi_{G,p,q}^\psi$ conditioned to match some configuration ω outside G' is exactly $\phi_{G',p,q'}^\xi$, where ξ describes the connections inherited from $\omega \cup \psi$ (two sites are wired if they are connected by a path in $\omega \cup \psi$ outside G' — see Lemma 4.13 in [Gri06]). This property is the direct analog of the DLR conditions for spin systems.

Comparison of boundary conditions when $q \geq 1$. An event is called *increasing* if it is preserved by addition of open edges. When $q \geq 1$, the model satisfies the FKG-inequality, or is *positively associated* (see Lemma 4.14 in [Gri06]), which has the following consequence: for any boundary conditions $\psi \leq \xi$ (meaning that ψ is finer than ξ , or in other words, that there are fewer connections in ψ than in ξ), we have

$$\phi_{G,p,q}^\psi(A) \leq \phi_{G,p,q}^\xi(A) \quad (2.2)$$

for any increasing event A . This last property, combined with the domain Markov property, provides a powerful tool to study the decorrelation between events.

Examples of boundary conditions: free, wired, Dobrushin. Three boundary conditions play a special role in the study of random-cluster models:

- The *wired* boundary conditions, denoted by $\phi_{G,p,q}^1$, is specified by the fact that all the vertices on the boundary are pairwise connected.
- The *free* boundary conditions, denoted by $\phi_{G,p,q}^0$, is specified by the absence of wirings between boundary sites.

These boundary conditions are extremal for stochastic ordering, since any boundary condition is smaller (resp. greater) than the wired (resp. free) boundary conditions.

- The *Dobrushin* boundary conditions: Assume now that ∂G is a self-avoiding polygon in \mathbb{L} , let a and b be two sites of ∂G . The triple (G, a, b) is called a *Dobrushin domain*. Orienting its boundary counterclockwise defines two oriented boundary arcs ∂_{ab} and ∂_{ba} ; the Dobrushin boundary conditions are defined to be free on ∂_{ab} (there are no wirings between boundary sites) and wired on ∂_{ba} (all the boundary sites are pairwise connected). These arcs are referred to as the *free arc* and the *wired arc*, respectively. The measure associated to these boundary conditions will be denoted by $\phi_{G,p,q}^{a,b}$ or simply $\phi_G^{a,b}$.

Infinite-volume measures and the definition of the critical point. The domain Markov property and comparison between boundary conditions allow us to define infinite-volume measures. Indeed, one can consider a sequence of measures on boxes of increasing sizes with free boundary conditions. This sequence is increasing in the sense of stochastic domination, which implies that it converges weakly to a limiting measure, called the random-cluster measure on \mathbb{Z}^2 with free boundary condition (and denoted by $\phi_{p,q}^0$). This classic construction can be performed with many other sequences of measures, defining several *a priori* different infinite-volume measures on \mathbb{Z}^2 . For instance, one can define the

random-cluster measure $\phi_{p,q}^1$ with wired boundary condition, by considering the decreasing sequence of random-cluster measures on finite boxes with wired boundary condition.

On \mathbb{Z}^d , for a given $q \geq 1$, it is known that uniqueness of the infinite-volume measure can fail only for p in a countable set \mathcal{D}_q , see Theorem 4.60 of [Gri06]. Since all limit measures are sandwiched between $\phi_{p,q}^0$ and $\phi_{p,q}^1$ w.r.t. stochastic domination, the countability of \mathcal{D}_q implies that there exists a *critical point* p_c such that for *any* infinite-volume measure with $p < p_c$ (resp. $p > p_c$), there is almost surely no infinite component of connected sites (resp. at least one infinite component).

Planar duality. In two dimensions, a random-cluster measure on a subgraph G of \mathbb{Z}^2 with free boundary conditions can be associated with a dual measure in a natural way. First define the *dual lattice* $(\mathbb{Z}^2)^*$, obtained by putting a vertex at the center of each face of \mathbb{Z}^2 , and by putting edges between nearest neighbors. The *dual graph* G^* of a finite graph G is given by the sites of $(\mathbb{Z}^2)^*$ associated with the faces adjacent to an edge of G . The edges of G^* are the edges of $(\mathbb{Z}^2)^*$ that connect two of its sites – note that any edge of G^* corresponds to an edge of G .

A dual model can be constructed on the dual graph as follows: for a percolation configuration ω , each edge of G^* is *dual-open* (or simply open), resp. *dual-closed*, if the corresponding edge of G is closed, resp. open. If the primal model is a random-cluster model with parameters (p, q) , then it follows from Euler’s formula (relating the number of vertices, edges, faces, and components of a planar graph) that the dual model is again a random-cluster model, with parameters (p^*, q^*) – in general, one must be careful about the boundary conditions. For instance, on a graph G , the random-cluster measure $\phi_{G,p,q}^0$ is dual to the measure ϕ_{G^*,p^*,q^*}^1 , where (p^*, q^*) satisfies

$$\frac{pp^*}{(1-p)(1-p^*)} = q \quad \text{and} \quad q^* = q.$$

Similarly, the dual of Dobrushin boundary conditions are Dobrushin boundary conditions themselves.

The critical point $p_c(q)$ of the model is the self-dual point $p_{\text{sd}}(q)$ for which $p = p^*$ (this has been recently proved in [BDC10a]), whose value can be derived:

$$p_{\text{sd}}(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}.$$

In the following, we need to consider connections in the dual model. Two sites x and y of G^* are said to be *dual-connected* if there exists a connected path of open dual-edges between them. Similarly to the primal model, we define *dual clusters* as maximal connected components for dual-connectivity.

FK-Ising model: crossing probabilities at criticality. For the value $q = 2$ of the parameter, the random-cluster model is related to the Ising model. In this case, the random-cluster model is now well-understood. The uniqueness of the infinite volume limit for all p is known since Onsager; see [Wer09b] for a short and elegant proof (or [DCS11, Proposition 3.10] for a version of Werner’s proof in English). The value $p_c = p_{\text{sd}}$ is implied by the computation by Kaufman and Onsager of the partition function of the Ising model, and an alternative proof has been proposed recently by Beffara and Duminil-Copin [BDC10b]. Moreover, in [Smi10], Smirnov proved conformal invariance of this model

at the self-dual point p_{sd} . As mentioned in the Introduction, criticality can sometimes be characterized by the fact that crossing probabilities are bounded uniformly away from 0 and 1. This fact was proved in [DCHN10] (our result is an extension of this one away from criticality):

Theorem 2.1 (RSW-type crossing bounds, [DCHN10]). *Let $0 < \rho_1 < \rho_2$. There exist two constants $0 < c_- \leq c_+ < 1$ (depending only on ρ_1 and ρ_2) such that for any rectangle R with side lengths n and $m \in [\rho_1 n, \rho_2 n]$ (i.e. with aspect ratio bounded away from 0 and ∞ by ρ_1 and ρ_2), one has*

$$c_- \leq \phi_{R,p_c,2}^\xi(\mathcal{C}_v(R)) \leq c_+$$

for any boundary conditions ξ .

In the rest of this section, we consider only random-cluster models on the two-dimensional square lattice with parameter $q = 2$, hence we drop the dependency on q in the notation. In this case, the model is called FK-Ising model.

2.2 Connectivity probabilities and the fermionic observable

The random-cluster with cluster-weight $q = 2$ is a model with long-range dependence. In particular, boundary conditions play a crucial role in connectivity probabilities. While general percolation arguments are sometimes sufficient to estimate connectivity probabilities in the bulk [BDC10a], there are very few possibilities to control probabilities in the presence of boundary conditions. We thus need a new argument to control these crossing probabilities.

When $q = 2$, Smirnov's fermionic observable provides us with a powerful tool to study such probabilities. In the next paragraph, we introduce the loop representation of the random-cluster model and we define Smirnov's observable. In the next one, we remind several properties of this observable at and away from the critical point. We list them without proof, since they are already presented in various places.

The medial lattice and the loop representation. Let G be a finite subgraph of \mathbb{Z}^2 together with Dobrushin boundary condition given by the boundary points a and b . Let G^* be the dual graph, with the natural definition that respects the boundary condition, see the left side of Figure 2.1. Declare *black* the sites of G and *white* the sites of G^* . Replace every site with a colored diamond, as in the right side of Figure 2.1. The *medial graph* $G_\diamond = (V_\diamond, E_\diamond)$ is defined as follows: E_\diamond is the set of diamond sides which belong to both a black and a white diamond; V_\diamond is the set of all the endpoints of the edges in E_\diamond . We obtain a subgraph of a rotated (and rescaled) version of the usual square lattice. We give G_\diamond an additional structure as an oriented graph by orienting its edges clockwise around white faces.

The random-cluster measure on (G, a, b) with Dobrushin boundary conditions has a rather convenient representation in this setting. Consider a configuration ω . It defines clusters in G and dual clusters in G^* . Through every vertex of the medial graph passes either an open bond of G or a dual open bond of G^* , hence there is a unique way to draw Eulerian (i.e., using every edge exactly once) loops on the medial lattice — *interfaces*, separating clusters from dual clusters. Namely, a loop arriving at a vertex of the medial lattice always makes a $\pi/2$ turn so as not to cross the open or dual open bond through this vertex, see Figure 2.1. Besides loops, the configuration contains a single curve joining the vertices adjacent to a and b , which are the only vertices in V_\diamond with three adjacent

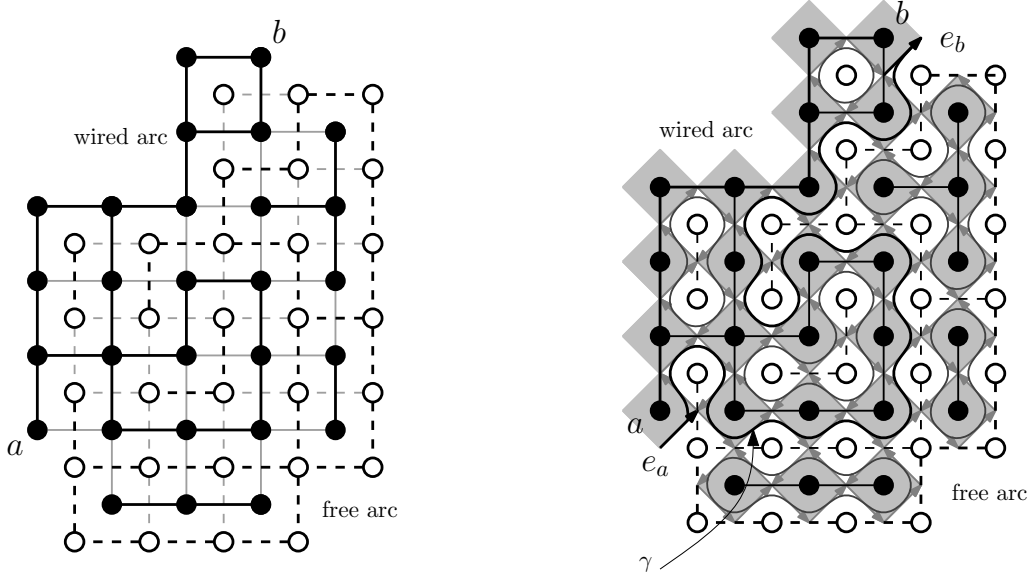


Figure 2.1: **Left:** A graph G with Dobrushin boundary conditions, and its dual G^* . The black (resp. white) sites are the sites of G (resp. G^*). The open bonds of G (resp. G^*) are represented by solid (resp. dashed) black bonds. **Right:** Construction of the medial lattice and the loop representation: the loops are interfaces between primal and dual clusters.

edges. This curve is called the *exploration path* and is denoted by γ . It corresponds to the interface between the cluster connected to the wired arc and the dual cluster connected to the free arc.

The first and last edge of γ are denoted by e_a and e_b , respectively. More generally, e_c denotes the medial edge pointing north-east and bordering the diamond associated to c .

This construction gives a bijection between random-cluster configurations on G and Eulerian loop configurations on G_\diamond . The probability measure can be nicely rewritten (using Euler's formula) in terms of the loop picture:

$$\phi_G^0(\omega) = \frac{x(p)^{\# \text{ open bonds }} \sqrt{2}^{\# \text{ loops }}}{\tilde{Z}(p, G)}, \quad \text{where} \quad x(p) := \frac{p}{(1-p)\sqrt{2}}$$

and $\tilde{Z}(p, G)$ is a normalizing constant. Notice that $p = p_c$ if and only if $x(p) = 1$. This bijection is called the *loop representation* of the random-cluster model. The orientation of the medial graph gives a natural orientation to the interfaces in the loop representation.

The fermionic observable. Fix a Dobrushin domain (G, a, b) . Following [Smi10], we now define an observable F on the edges of its medial graph, *i.e.* a function $F : E_\diamond \rightarrow \mathbb{C}$. Roughly speaking, F is a modification of the probability that the exploration path passes through a given edge.

First, the *winding* $W_\Gamma(z, z')$ of a curve Γ between two edges z and z' of the medial graph is the total rotation (in radians and oriented counter-clockwise) that the curve makes from the mid-point of edge z to that of edge z' . We define the observable $F = F_p$ for any edge $e \in E_\diamond$ as

$$F(e) := \phi_{p,2,G}^{a,b} \left(e^{\frac{i}{2} W_\gamma(e, e_b)} \mathbb{1}_{e \in \gamma} \right), \quad (2.3)$$

where γ is the exploration interface from a to b .

Relation with connectivity probabilities. As was mentioned earlier, the fermionic observable is related to connectivity properties of the model via the following fact:

Lemma 2.2 (Equation (14) in [Smi10], Lemma 2.2 in [BDC10b]). *Let $u \in G$ be a site next to the free arc, and e be a side of the black diamond associated to u which borders a white diamond of the boundary. Then,*

$$|F(e)| = \phi_{p,2,G}^{a,b}(u \leftrightarrow \text{wired arc}). \quad (2.4)$$

Integrability relations of the fermionic observable. A lot of information has been gathered on the fermionic observable during the last few years. In particular, it satisfies local relations that allow to determine its scaling limit.

Proposition 2.3 (Lemma 2.3 of [BDC10b]). *Consider a medial vertex v in $G^\diamond \setminus \partial G^\diamond$. We index the two edges pointing towards v by A and C , and the two others by B and D , such that the alphabetical order is clockwise oriented. Then,*

$$F(A) - F(C) = e^{i\alpha} i[F(B) - F(D)]. \quad (2.5)$$

where

$$e^{i\alpha} := \frac{e^{i\pi/4} + x}{e^{i\pi/4}x + 1}. \quad (2.6)$$

Moreover, the complex argument modulo π of F at any edge e is determined by the direction of e . More precisely, if e points in the same direction as e_b , then F is real. Similarly, F belongs to $e^{-i\pi/4}\mathbb{R}$ (resp. $i\mathbb{R}$, $e^{i\pi/4}\mathbb{R}$) if e makes an angle of $\pi/2$ (resp. π , $3\pi/2$) with the edge e_b , see Fig. 2.2. Knowing the complex argument modulo π , together with (2.5), allow one to express the value of the observable at one edge e in terms of the values at two edges incident to one of the endpoints of e . This important fact was used extensively in [Smi10] and in any following work since. For instance, it implies that F is determined by the relations (2.5) and the fact that $F(e_b) = 1$ and $F(e_a) = e^{iW_\Gamma(e_a, e_b)/2}$, where Γ is any path from e_a to e_b staying in G^\diamond . In addition, these relations have a very special form. In particular, it has been proved for $x = 1$ in [Smi10] and then extended in [BDC10b] (Lemma 4.4) to general values of x that F is massive harmonic inside the domain:

Proposition 2.4. *Let $p \in (0, 1)$ and X with four neighbors in $G \setminus \partial G$, we have*

$$\Delta_p F(e_X) = 0, \quad (2.7)$$

where the operator Δ_p is defined by

$$\Delta_p g(e_X) := \frac{\cos[2\alpha]}{4} \left(\sum_{Y \sim X} g(e_Y) \right) - g(e_X). \quad (2.8)$$

Observe that $\alpha(p) = 0$ if and only if $p = p_c$. In this case, the observable is discrete harmonic inside the domain. As mentioned before, this is one of the main ingredients of Smirnov's proof of conformal invariance: when properly rescaled, the observable converges to a harmonic map. Boundary conditions for F correspond to discretizations of the

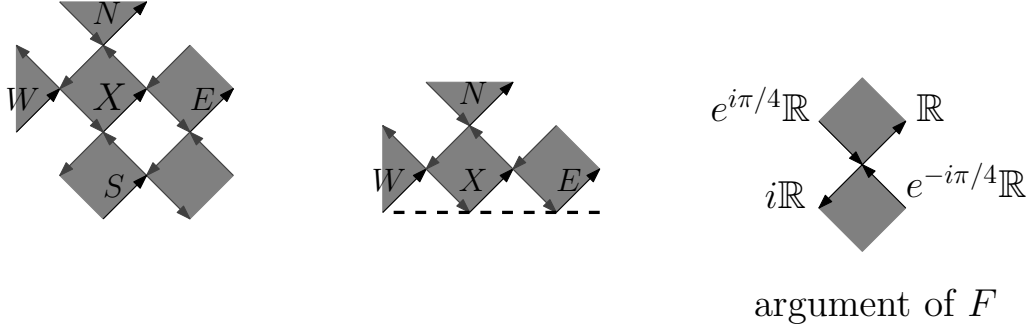


Figure 2.2: **Left:** An edge inside the domain: it has four edges oriented the same way at distance two. **Right:** An edge on the free arc with the associated indexation.

Riemann-Hilbert problem. These boundary conditions are quite complicated to study at a discrete level, and Smirnov used a discrete primitive H of (the imaginary part of) F^2 to handle them. The function H was then solving a discretized Dirichlet problem.

In particular, the use of H made the estimation of F on the free arc possible. More precisely, F was related to the square root of modified harmonic measures (see Proposition 3.2 of [DCHN10]). This fact was crucial in the proof of Theorem 2.1 [DCHN10]. Without entering into details, let us say that in Dobrushin “domains” (G, a, b) , the probability at criticality for a site x on the free arc to be connected to the wired arc (which is the modulus of the observable, thanks to Lemma 2.2) is on the order of the square-root of the harmonic measure of the wired arc seen from x . Equivalently, for a dual site u on the wired arc, the probability of being dual-connected to the free arc is of order of the square-root of the harmonic measure of the free arc seen from u . We refer to [DCHN10] for additional details on these facts.

We will be using this fact for two nice infinite Dobrushin domains:

- The infinite strip $S_n = \mathbb{Z} \times [0, n]$ of height n . Denote by $\phi_{S_n, p}^{-\infty, \infty}$ the random-cluster measure with parameter p , free boundary conditions on the bottom and wired boundary conditions on the top. The probability at criticality for a dual-site on the top to be dual-connected to the free arc is of order $1/\sqrt{n}$ (since the harmonic measure of the free arc is $1/n$, via the Gambler’s ruin).
- The upper half-plane \mathbb{H} . Denote by $\phi_{\mathbb{H}, p}^{0, \infty}$ the random-cluster measure with parameter p , free boundary conditions on $\mathbb{Z}_+ = \{0, 1, \dots\}$ and wired boundary conditions on $\mathbb{Z}_- = \{\dots, -2, -1, 0\}$. The probability at criticality for the dual site adjacent to $-n$ to be dual-connected to the free arc is of order $1/\sqrt{n}$ for the same reason as for the strip.

When $p \leq p_c$, the fermionic observable F_p can be defined in these two domains even though they are infinite (see [BDC10b]). In the strip, γ goes from $-\infty$ to ∞ , while in \mathbb{H} , it goes from 0 to ∞ . One should be careful about the definition of the winding since e_b does not make sense: the winding is fixed to be equal to 0 on edges of the free arc pointing north-east. Since the observable in infinite volume is the limit of observables in finite volume, it still satisfies the properties of the previous section.

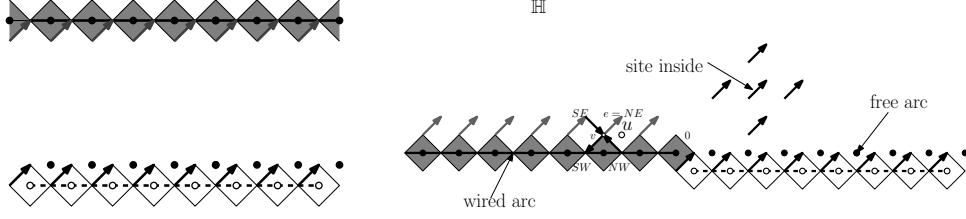


Figure 2.3: **Left:** The strip. **Right:** The upper half-plane. In this case, τ is the hitting time of grey edges.

2.3 Integrability relations of the fermionic observable away from criticality

Away from the critical point, the primitive H is not available anymore. Nevertheless, F_p is massive harmonic inside the domain. In fact, F_p satisfies very explicit relations on the free arc of the domain, as well. (On the wired arc, such relations are not available.) Indeed, Case 3 of Lemma 4.4 of [BDC10b] says that

$$\begin{aligned} \Delta_p F_p(e_X) &= \frac{\cos 2\alpha}{2(1 + \cos(\pi/4 - \alpha))} [F_p(e_W) + F_p(e_N)] + \frac{\cos(\pi/4 + \alpha)}{1 + \cos(\pi/4 - \alpha)} F_p(e_E) - F_p(e_X) \\ &= 0 \end{aligned} \quad (2.9)$$

for X on the free arc (excluding 0 in the case of the upper half-plane). When $p = p_c$, the sum of the coefficients on the right equals 0, which means that the observable has an interpretation in terms of reflected random walks. This relates to the discretization of the Riemann-Hilbert boundary problem, and it provides an alternative strategy to handle the scaling limit of the observable.

Away from criticality, we can also interpret these relations in terms of a random process. Define the Markov process with generator Δ_p , which one can interpret as the random walk of a massive particle. We write this process $(X_n^{(p)}, m_n^{(p)})$ where $X_n^{(p)}$ is a random walk with jump probabilities defined in terms of Δ_p — the proportionality between jump probabilities is the same as the proportionality between coefficients — and $m_n^{(p)}$ is the mass associated to this random walk. The law of the random walk starting at an edge x is denoted \mathbb{P}_p^x . In order to simplify the notation, we drop the dependency in p in $(X_n^{(p)}, m_n^{(p)})$ and simply write (X_n, m_n) . Note that the mass of the walk decays by a factor $\cos 2\alpha$ at each step inside the domain, and by some other factor on the free arc.

Define τ to be the hitting time of the wired arc, more precisely, of the set ∂ of medial edges pointing north-east and having one end-point on the wired arc (the grey edges in Fig. 2.3). The fact that $\Delta_p F_p = 0$ for every edge $x \notin \partial$ implies for any $t \geq 0$ that

$$F_p(x) = \mathbb{E}_p^x[F_p(X_{t \wedge \tau})m_{t \wedge \tau}]. \quad (2.10)$$

Since $m_\tau \leq 1$, $F_p(X_{t \wedge \tau})m_{t \wedge \tau}$ is uniformly integrable and (2.10) can be improved to

$$F_p(x) = \mathbb{E}_p^x[F_p(X_\tau)m_\tau]. \quad (2.11)$$

This will be the principal tool in our study.

Proposition 2.5. *Let $\lambda > 0$. There exists $C_1 = C_1(\lambda)$ such that for every $n > 0$ and $p_c \geq p > p_c - \frac{\lambda}{n}$,*

$$\phi_{S_{n,p}}^{-\infty, \infty}(0 \leftrightarrow in + \mathbb{Z}) \geq \frac{C_1}{\sqrt{n}}. \quad (2.12)$$

There exists $C_2 = C_2 > 0$ such that for every $n > 0$ and $p < p_c - \frac{C_2 \sqrt{\log n}}{n}$,

$$\phi_{S_{n,p}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) \leq \frac{C_2}{n^4}. \quad (2.13)$$

Proof. In both cases, we study the probability for a point on the free arc to be connected to the wired arc. In particular, Lemma 2.2 implies that the quantities on the left of (2.12) and (2.13) are equal to $|F(e_0)|$ (or $F(e_0)$ in this case, since the winding is fixed on the boundary). Moreover, (2.11) allows us to write

$$\phi_{S_{n,p}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) = F(e_0) = \mathbb{E}_p^0[F_p(X_\tau)m_\tau]$$

Let us first deal with (2.12). Note that $F_p(X_\tau) = F_p(in) = F_{p^*}(e_0)$, where p^* is the dual parameter. Hence

$$\begin{aligned} \phi_{S_{n,p}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) &= \mathbb{E}_p^0[F_{p^*}(e_0)m_\tau] \\ &= \phi_{S_{n,p^*}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) \mathbb{E}_p^0[m_\tau], \end{aligned}$$

using Lemma 2.2 again. Since $p > p_c - \frac{\lambda}{n}$, $\cos 2\alpha = m_1^{(p)}$ is larger than $1 - c(\lambda/n)^2$ for some $c > 0$. Using the upper tail of τ/n^2 , we deduce that

$$\mathbb{E}^x[m_\tau] \geq C$$

for some $C = C(\lambda)$. In addition to this,

$$\phi_{S_{n,p^*}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) \geq \frac{C}{\sqrt{n}},$$

where we used the standard estimate of the probability at criticality, together with $p^* > p_c$. The two inequalities together yield (2.12).

Let us now turn to (2.13). When $p < p_c - C\sqrt{\log n}/n$, we use the expansion of α near p_c and $\cos 2\alpha \leq 1 - c(\log n)/n^2$ (for some constant $c = c(C)$) to deduce

$$\phi_{S_{n,p}}^{-\infty,\infty}(0 \leftrightarrow in + \mathbb{Z}) \leq \mathbb{E}_p^0[m_\tau] \leq \mathbb{E}_p^0[(1 - c_2(\log n)/n^2)^\tau] \approx n^{-c'}$$

for $c' = c'(C)$. In order to conclude, c' can be chosen larger than 4 by tuning C . \square

The previous proof of (2.12) was based on a comparison with the estimates at criticality: when $p > p_c - \lambda/n$ the connection probabilities are of the same order as the critical ones. We push this reasoning further in the following proposition.

Proposition 2.6. *For any $\lambda > 0$, there exists $C_3 = C_3(\lambda) > 0$ such that for every $n > 0$ and $p > p_c - \frac{\lambda}{n}$,*

$$\phi_{\mathbb{H},p}^{0,\infty}(n \leftrightarrow \mathbb{Z}_-) \geq \frac{C_3}{\sqrt{n}}. \quad (2.14)$$

Let us first prove an easy yet slightly technical result. It should be compared to Lemma 2.2.

Lemma 2.7. *Let u be a dual vertex adjacent to the wired arc of \mathbb{H} ,*

$$F_p(e_u) \asymp \phi_{\mathbb{H},p}^{0,\infty}(u \overset{*}{\leftrightarrow} \mathbb{Z}_+),$$

where e_u is the edge pointing north-east and adjacent to u , and \asymp means that the ratio is uniformly bounded away from 0 and ∞ .

Proof. If v is the vertex of the medial lattice on the left of u , relation (2.5) around v gives $F(NW) + F(SE) = e^{i\alpha}(F(NE) + F(SW))$, where edges are indexed with respect to the direction **they are pointing to** (see Fig. 2.3). Since we know the complex argument modulo π of the observable, we can project the relation on $e^{-i\pi/4}\mathbb{R}$, to find

$$e^{i\pi/4}F(NW) - \cos(\pi/4 - \alpha)iF(SW) = \cos(\pi/4 + \alpha) F(NE).$$

Now, the argument of the observable at NW and SW is in fact determined, since the winding on the boundary is deterministic (it equals $-\pi/2$ for NW , and $-\pi$ for SW). Therefore, Lemma 2.2 implies

$$\begin{aligned} e^{i\pi/4}F(NW) &= |F(NW)| = \phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+) \\ iF(SW) &= |F(SW)| = \phi_{\mathbb{H},p}^{0,\infty}(u - 1 \xleftrightarrow{*} \mathbb{Z}_+). \end{aligned}$$

So, using the fact that $NE = e_u$, we get

$$\cos(\pi/4 + \alpha)F(e_u) = \phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+) - \cos(\pi/4 - \alpha)\phi_{\mathbb{H},p}^{0,\infty}(u - 1 \xleftrightarrow{*} \mathbb{Z}_+).$$

Now, $\phi_{\mathbb{H},p}^{0,\infty}(u - 1 \xleftrightarrow{*} \mathbb{Z}_+) \leq \phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+)$ thanks to the comparison between boundary conditions. We deduce

$$\frac{1 - \cos(\pi/4 - \alpha)}{\cos(\pi/4 + \alpha)} \phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+) \leq F(e_u) \leq \frac{1}{\cos(\pi/4 + \alpha)} \phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+)$$

which is the claim. \square

We are now in a position to prove the proposition.

Proof of Proposition 2.6. Fix $n > 0$ and $p \geq p_c - \frac{\lambda}{n}$ and denote the fermionic observable in $(\mathbb{H}, 0, \infty)$ by F_p . Lemma 2.7 implies

$$F_p(n) = \mathbb{E}_p^n[F_p(X_\tau)m_\tau] \asymp \mathbb{E}_p^n[\phi_{\mathbb{H},p}^{0,\infty}(X_\tau \xleftrightarrow{*} \mathbb{Z}_+)m_\tau]. \quad (2.15)$$

We know that

$$\phi_{\mathbb{H},p}^{0,\infty}(u \xleftrightarrow{*} \mathbb{Z}_+) \geq C_3/\sqrt{|u|},$$

hence (2.15) implies

$$F_p(n) \geq C_4 \mathbb{E}_p^n[\phi_{\mathbb{H},p}^{0,\infty}(X_\tau \xleftrightarrow{*} \mathbb{Z}_+) m_\tau] \geq C_4 C_3 \mathbb{E}_p^n[m_\tau^n / \sqrt{|X_\tau|}], \quad (2.16)$$

with two universal constants $C_3, C_4 > 0$.

Therefore, it is sufficient to prove that $|X_\tau|$ is not larger than n and that m_τ is larger than some constant ε with probability bounded away from 0 uniformly in n . The second condition can be replaced by the event $\tau \leq n^2$ for instance.

The walk X_t away from the real axis is just simple random walk, while on the free arc it has some outwards drift. So, it is sufficient to prove that X_t exits $[0, 2n] \times [0, n]$ through $[0, 2n] \times \{n\}$ in fewer than $n^2/2$ steps with probability larger than some constant $c > 0$ not depending on n . Indeed, it has then a uniformly positive probability to exit the domain in fewer than $n^2/2$ additional steps and to satisfy $|X_\tau| \leq n$.

Consider $(X_t)_{t \leq n^2/2} = (A_t, B_t)_{t \leq n^2/2}$ conditioned on the event that $(X_t)_{t \leq n^2/2}$ visits the free arc fewer than n times. The probability that the first coordinate is less than n for every $t \leq n^2/2$ is bounded away from 0 uniformly in n (since the number of visits of (X_t) to the free arc is less than n , (A_t) can be compared to a symmetric random walk with a deterministic drift of order rn for $r < 1$). Now, conditioned on the visits of (X_t) to the free arc, (A_t) and (B_t) are independent. Thus, (B_t) is a random walk reflected at the origin conditioned on the fact that it does not visit 0 more than n times. In time $n^2/2$, it reaches height n with probability bounded away from 0, uniformly in n . The claim follows. \square

2.4 Proof of Theorem 1.3

We first prove crossing probabilities in rectangles with specific boundary conditions. Then, we use these crossings to construct crossings in arbitrary rectangles with free boundary conditions.

Crossing in rectangles with Dobrushin boundary conditions. Let us first use the estimates obtained in the previous subsection to prove crossing probabilities in the strip and the half-plane. The proof follows a second moment argument.

Proposition 2.8. *Fix $\lambda > 0$. There exists $C_6 = C_6(\lambda) > 0$ such that for every $n > 0$ and $p > p_c - \frac{\lambda}{n}$,*

$$\phi_{S_n,p}^{-\infty,\infty}([-n,n] \leftrightarrow in + \mathbb{Z}) \geq C_6$$

and

$$\phi_{\mathbb{H},p}^{-\infty,\infty}([3n,4n] \leftrightarrow \mathbb{Z}_-) \geq C_6.$$

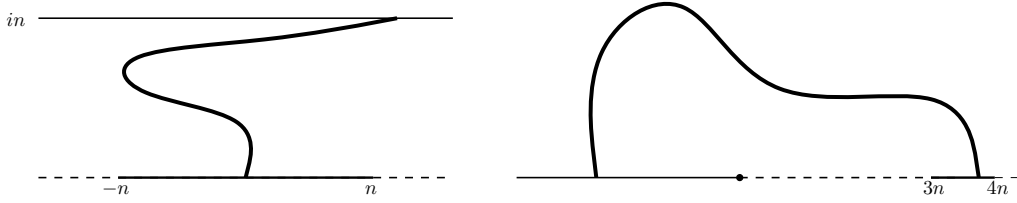


Figure 2.4: The two crossing events of Proposition 2.8.

Proof. We present the proof for S_n (a similar argument works for \mathbb{H}). Let N be the (random) number of sites on $[-n,n]$ which are connected by an open path to $in + \mathbb{Z}$. Proposition 2.5 implies that

$$\phi_{S_n,p}^{-\infty,\infty}(N) = \sum_{x \in [-n,n]} \phi_{S_n,p}^{-\infty,\infty}(x \leftrightarrow in + \mathbb{Z}) \geq (2n+1) \frac{C_1}{\sqrt{n}} \geq 2C_1\sqrt{n}. \quad (2.17)$$

Moreover, for $p \leq p_c$,

$$\phi_{S_n,p}^{-\infty,\infty}(N^2) \leq \phi_{S_n,p_c}^{-\infty,\infty}(N^2).$$

The right hand side is a quantity at the critical point and was already studied in the proof of the main theorem of [DCHN10] (in fact, only closely related quantities were studied, but the generalization is straightforward). In particular, it was proved in that article that

$$\phi_{S_n,p_c}^{-\infty,\infty}(N^2) \leq C_6 n.$$

Cauchy-Schwarz thus implies that

$$\phi_{S_n,p}^{-\infty,\infty}([-n,n] \leftrightarrow in + \mathbb{Z}) = \phi_{S_n,p}^{-\infty,\infty}(N > 0) \geq 4C_1^2/C_6,$$

uniformly in n . For $p > p_c$ the result follows from monotonicity. \square

It is now easy to reduce crossing probabilities in the strip and the half-plane to crossing probabilities in (possibly very large) rectangles. The idea is that a crossing cannot explore too much of the strip or the half-plane, since there exist slightly supercritical dual crossings preventing it.

Proposition 2.9. *Fix $\lambda > 0$. There exist $C_7 > 0$ and $M > 0$ such that for every $n > 0$ and $p > p_c - \frac{\lambda}{n}$,*

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}([-n, n] \leftrightarrow in + \mathbb{Z}) \geq C_7$$

and

$$\phi_{[-Mn, Mn] \times [0, Mn], p}^{0, -Mn}([3n, 4n] \leftrightarrow \mathbb{Z}_-) \geq C_7.$$

Proof. As before, we do this in the case of the strip. Fix M large enough so that, at criticality, the probability that there exists a vertical dual crossing with free boundary conditions of $[n, Mn] \times [0, n]$ exceeds $1 - C_6/3$ (use Theorem 2.1 to prove this fact). Then, with probability $C_6/3$, there will exist a crossing of $[-n, n]$ to $in + \mathbb{Z}$ and two dual vertical crossings in $[n, Mn] \times [0, n]$ and $[-Mn, -n] \times [0, n]$. The domain Markov property and the comparison between boundary conditions imply the result. \square

Crossing in rectangles with free boundary conditions. A consequence of Proposition 2.9 is the existence of crossings inside a box with free boundary conditions everywhere. Indeed, although the previous result only deals *a priori* with domains where a part of the boundary is already wired, this condition can be removed.

Proposition 2.10. *Fix $\lambda > 0$. There exist $C_8, M > 0$ such that for every $n > 0$ and $p > p_c - \frac{\lambda}{n}$,*

$$\phi_{[-Mn, Mn] \times [0, n], p}^0([-Mn, Mn] \times [0, n/2] \text{ is crossed vertically}) \geq C_8.$$

Proof. Fix M so that Proposition 2.9 holds true. Let A_n be the event that $[-Mn, Mn] \times [0, n/2]$ is crossed vertically. We have for every $n > 0$,

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(A_n) \geq C_7.$$

Let B_n be the event that $[-Mn, Mn] \times [n/2, n]$ is dual-crossed horizontally. Theorem 2.1 implies that

$$\phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(B_n | A_n) \geq c$$

for some constant $c > 0$, uniformly in n and $p < p_c$. Now,

$$\begin{aligned} \phi_{[-Mn, Mn] \times [0, n], p}^0(A_n) &\geq \phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(A_n | B_n) \\ &\geq \phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(A_n \cap B_n) \\ &= \phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(B_n | A_n) \cdot \phi_{[-Mn, Mn] \times [0, n], p}^{(i-M)n, (i+M)n}(A_n) \\ &\geq c \cdot C_7. \end{aligned} \quad \square$$

We now prove that crossings of rectangles of any aspect ratio also exist.

Lemma 2.11. *Fix $\lambda > 0$ and $\kappa > 0$. There exists $C_9 = C_9(\kappa, \lambda) > 0$ such that for every n and $p > p_c - \frac{\lambda}{n}$,*

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([0, \kappa n] \times [0, n] \text{ is crossed horizontally}) \geq C_9.$$

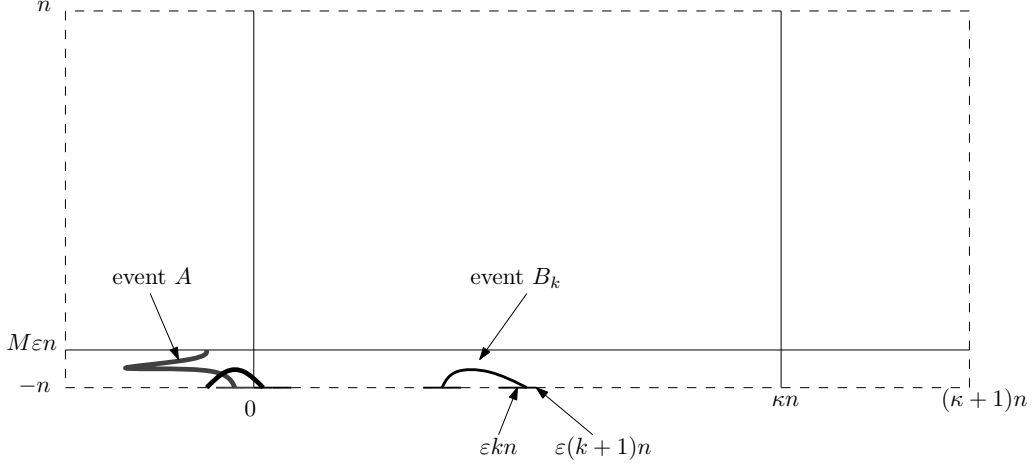


Figure 2.5: The intersections of the events A and B_k create a crossing of the rectangle $[-n, n] \times [0, \kappa n]$.

Proof. Fix $M = M(\lambda)$ as in Propositions 2.9 and 2.10. Let $\varepsilon = 1/(2M)^2$. Let A be the event that there exists a crossing from $[-\varepsilon n, \varepsilon n]$ to $iM\varepsilon n + \mathbb{Z}$, and let B_k be the event that there exists a path in $\mathbb{Z} \times [0, M\varepsilon n]$ from $[(k+1)\varepsilon n, (k+2)\varepsilon n]$ to $[(k-1)\varepsilon n, k\varepsilon n]$. See Figure 2.5. We have

$$\begin{aligned} \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([-n, n] \times [0, \kappa n] \text{ is crossed horizontally}) \\ \geq \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0 \left(A \cap \bigcap_{k=0}^{\kappa/\varepsilon-1} B_k \right) \\ = \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(A) \prod_{k=0}^{\kappa/\varepsilon-1} \phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(B_k | A, B_r, r < k). \end{aligned}$$

Furthermore,

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0(A) \geq \phi_{[-n, n] \times [0, n/(2M)], p}^0(A).$$

Now, since $M\varepsilon = 1/(4M)$, the event A in $[-n, n] \times [0, n/(2M)]$ corresponds to the existence of a crossing from the bottom to the middle, but with the additional constraint that it starts between $[-\varepsilon n, \varepsilon n]$. A union bound, comparison between boundary conditions, and Proposition 2.10 imply that

$$\phi_{[-n, n] \times [0, n/(2M)], p}^0(A) \geq \varepsilon \phi_{[-n, n] \times [0, n/(2M)], p}^0([-n, n] \times [0, n/(4M)]) \geq \varepsilon C_8.$$

Furthermore,

$$\phi_{[-n, n] \times [-n, (\kappa+1)n], p}^0(B_k | A, B_r, r < k) \geq \phi_{[(k-M)\varepsilon n, (k+M)\varepsilon n] \times [0, M\varepsilon n], p}^{k\varepsilon n, \infty}(B_k) \geq C_7,$$

using the comparison between boundary conditions and the half-plane case of Proposition 2.9. Altogether, we obtain that

$$\phi_{[-n, (\kappa+1)n] \times [0, n], p}^0([-n, n] \times [0, \kappa n] \text{ is crossed horizontally}) \geq C_8 C_7^{\kappa/\varepsilon},$$

and the lemma is proved. \square

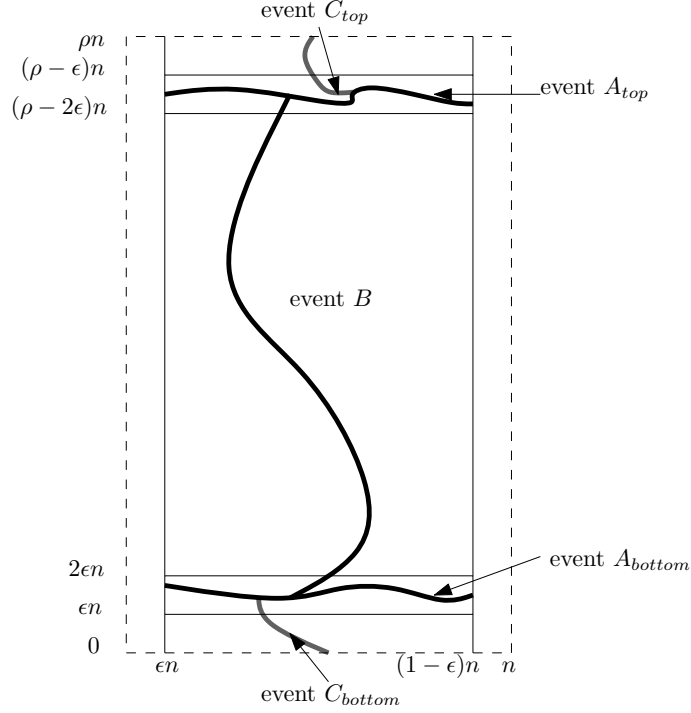


Figure 2.6: The five events involved in the proof of Theorem 1.3.

Proof of Theorem 1.3 Fix $\varepsilon < 1/(4M)$. Let A_{bottom} and A_{top} be the events that $[\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon n, 2\varepsilon n]$ and $[\varepsilon n, (1 - \varepsilon)n] \times [(\rho - 2\varepsilon)n, (\rho - \varepsilon)n]$ are crossed horizontally. Let B be the event that $[\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon n, (\rho - \varepsilon)n]$ is crossed vertically. Let C_{bottom} and C_{top} be the events that $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$ and $[\varepsilon n, (1 - \varepsilon)n] \times [(\rho - 2\varepsilon)n, \rho n]$ are crossed vertically. See Figure 2.6. By Lemma 2.11, the events A_{bottom} , A_{top} and B have probability bounded away from 0 uniformly in n . The FKG inequality implies that their intersection also has this property. Now, conditionally on A_{bottom} , the event C_{bottom} has probability larger than the probability that there exists a crossing in $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$ with wired boundary condition on the top and free boundary condition on the bottom. Proposition 2.9 implies that this probability is larger than C_7 since $(1 - 2\varepsilon)/(2\varepsilon) > 2M$ (the important thing is that the rectangle $[\varepsilon n, (1 - \varepsilon)n] \times [0, 2\varepsilon n]$ is wide enough). The same reasoning can be applied to C_{top} , ergo the claim follows. \square

2.5 Proofs of Theorems 1.2 and 1.4

First of all, a standard reasoning described in Step 2 of the proof of Theorem 1 of [BDC10b] shows that equation (2.13) implies the following lemma:

Lemma 2.12. *There exists $C_{10} > 0$ such that*

$$\phi_p(0 \leftrightarrow \partial[-n, n]^2) \leq C_{10}n^{-3} \quad (2.18)$$

for every n large enough and every $p \leq p_c - C_{10} \frac{\sqrt{\log n}}{n}$. \square

Proof of Theorem 1.2. Fix $C_{10} > 0$ as defined in Lemma 2.12. Theorem 1.3 implies the lower bound trivially. For the upper bound, it suffices to show that for any $\kappa > 0$,

$$\phi_{[-n,n] \times [-\kappa n, \kappa n], p}^1([-n, n] \times [-\kappa n, \kappa n] \text{ is crossed horizontally}) \rightarrow 0$$

whenever $(n, p) \rightarrow (\infty, 0)$ with $p \leq p_c - C_{10}\sqrt{\log n}/n$. Fix $\varepsilon > 0$ and $\kappa > 0$.

Take some $\delta > 0$ to be fixed later. Let A_n^{top} be the event that $[-(1-\delta)n, (1-\delta)n] \times [-\kappa n, (\kappa-2\delta)n]$ is crossed horizontally, and A_n^{bottom} be the event that $[-(1-\delta)n, (1-\delta)n] \times [-(\kappa-2\delta)n, \kappa n]$ is crossed horizontally. Furthermore, let B_n be the event that $[-(1-\delta)n, (1-\delta)n] \times [-(\kappa-\delta)n, (\kappa-\delta)n]$ contains a cluster of diameter δn . Notice that if the rectangle $[-n, n] \times [-\kappa n, \kappa n]$ is crossed horizontally, then A_n^{top} or A_n^{bottom} or B_n occurs.

Theorem 2.1 implies the existence of $\delta > 0$ such that the probability of A_n^{top} (and similarly for A_n^{bottom}) with wired boundary conditions is smaller than $\varepsilon/3$ for any $p < p_{sd}$ and $n > 0$. This will be our δ .

Define C_n to be the event that the annulus

$$S_n := [-n, n] \times [-\kappa n, \kappa n] \setminus [-(1-\delta)n, (1-\delta)n] \times [-(\kappa-\delta)n, (\kappa-\delta)n]$$

contains a closed circuit surrounding the inner box. Note that there exists $\eta > 0$ such that

$$\phi_{p, S_n}^1(C_n) \geq \eta,$$

thanks to Theorem 2.1 again. Since C_n is decreasing and B_n depends only on edges inside $[-(1-\delta)n, (1-\delta)n] \times [-(\kappa-\delta)n, (\kappa-\delta)n]$, we obtain

$$\phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^1(C_n | B_n) \geq \phi_{p, S_n}^1(C_n) \geq \eta.$$

Therefore,

$$\begin{aligned} \eta \phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^1(B_n) &\leq \phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^1(B_n \cap C_n) \\ &\leq \phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^1(B_n | C_n) \\ &\leq \phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^0(B_n). \end{aligned}$$

By a union bound, Lemma 2.12 and the definition of C_{10} imply that

$$\phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^0(B_n) \rightarrow 0 \quad \text{when } n \rightarrow \infty. \quad (2.19)$$

Therefore, $\phi_{p, [-n, n] \times [-\kappa n, \kappa n]}^1(B_n) \rightarrow 0$.

Summarizing, each of A_n^{top} , A_n^{bottom} and B_n has probability less than $\varepsilon/3$ for n large enough, which concludes the proof. \square

Let us now turn to the proof of Theorem 1.4. We have just proved that, for $\rho > 0$ and $\varepsilon > 0$, there exists $c = c(\varepsilon, \rho)$ such that for any $n \geq \frac{c}{p_c - p} \sqrt{\log \frac{1}{p_c - p}}$,

$$\phi_{p, [0, n] \times [0, \rho n]}^1(\mathcal{C}_h([0, n] \times [0, \rho n])) \leq \varepsilon.$$

The next lemma asserts that crossing probabilities in fact converge to 0 very quickly as soon as n is larger than the correlation length.

Lemma 2.13. *For any $p < p_c$, there exists $L(p)$ such that*

$$\frac{c}{p_c - p} \leq L(p) \leq \frac{1}{c(p_c - p)} \sqrt{\log \frac{1}{p_c - p}}$$

and

$$\phi_{p, [0, 2^k L(p)] \times [0, 2^{k+1} L(p)]}^1 \left(\mathcal{C}_h([0, 2^k L(p)] \times [0, 2^{k+1} L(p)]) \right) \leq e^{-2^k}$$

for any $k \geq 0$.

Proof. For $n > 0$, let

$$u_n := \max \left\{ \phi_{p, [0, n] \times [0, 2n]}^1 \left(\mathcal{C}_h([0, n] \times [0, 2n]) \right), \phi_{p, [0, n]^2}^1 \left(\mathcal{C}_h([0, n]^2) \right) \right\}.$$

We are going to show that

$$u_{2n} \leq 25 u_n^2. \quad (2.20)$$

First, cutting vertically the domain $[0, 2n]^2$ into two rectangles, together with comparison between boundary conditions, implies that

$$\phi_{p, [0, 2n]^2}^1 \left(\mathcal{C}_h([0, 2n]^2) \right) \leq \phi_{p, [0, n] \times [0, 2n]}^1 \left(\mathcal{C}_h([0, n] \times [0, 2n]) \right)^2 \leq u_n^2. \quad (2.21)$$

Second, cutting vertically the domain $[0, 2n] \times [0, 4n]$ into two, together with comparison between boundary conditions again, implies that

$$\phi_{p, [0, 2n] \times [0, 4n]}^1 \left(\mathcal{C}_h([0, 2n] \times [0, 4n]) \right) \leq \phi_{p, [0, n] \times [0, 4n]}^1 \left(\mathcal{C}_h([0, n] \times [0, 4n]) \right)^2.$$

Now, consider the rectangles

$$\begin{aligned} R_1 &:= [0, n] \times [0, 2n] \\ R_2 &:= [0, n] \times [n, 3n] \\ R_3 &:= [0, n] \times [2n, 4n] \\ R_4 &:= [0, n] \times [n, 2n] \\ R_5 &:= [0, n] \times [2n, 3n] \end{aligned}$$

These rectangles have the property that whenever $[0, n] \times [0, 4n]$ is crossed horizontally, at least one of the rectangles R_i is crossed (in the horizontal direction for R_1, R_2 and R_3 , and vertically otherwise). We deduce, using the comparison between boundary conditions, that

$$\phi_{p, [0, n] \times [0, 4n]}^1 \left(\mathcal{C}_h([0, n] \times [0, 4n]) \right) \leq 5 u_n,$$

and hence

$$\phi_{p, [0, 2n] \times [0, 4n]}^1 \left(\mathcal{C}_h([0, 2n] \times [0, 4n]) \right) \leq (5 u_n)^2. \quad (2.22)$$

Combining (2.21) and (2.22), we obtain (2.20). Iterating that, we easily obtain that, for every $k \geq 0$,

$$25 u_{2^k n} \leq (25 u_n)^{2^k}.$$

By Theorem 1.2, if $p < p_c$ and $n \geq \frac{c^{-1}}{p_c - p} \sqrt{\log \frac{1}{p_c - p}}$, where $c = \min\{c(1/100, 2), c(1/100, 1)\}$, then u_n satisfies

$$25 u_n \leq 1/e.$$

Therefore, the lemma follows for $L(p) = \frac{c^{-1}}{p_c - p} \sqrt{\log \frac{1}{p_c - p}}$. \square

Proof of Theorem 1.4. Fix $p > p_c$. Let

$$R_k := [0, L(p)2^k] \times [-L(p), L(p)(2^{k+1} - 1)] \quad \text{if } k \text{ is even,}$$

and

$$R_k := [0, L(p)2^{k+1}] \times [-L(p), L(p)(2^k - 1)] \quad \text{if } k \text{ is odd.}$$

Define E_k to be the event that R_k is crossed in the long direction. The FKG inequality implies that

$$\begin{aligned} \phi_p^0(0 \leftrightarrow \infty) &\geq \phi_p^0(0 \leftrightarrow \{L(p)\} \times [-L(p), L(p)]) \cdot \prod_{k \geq 0} \phi_p^0(E_k) \\ &\geq \frac{1}{4} \phi_p^0(0 \leftrightarrow \partial[-L(p), L(p)]^2) \cdot \prod_{k \geq 0} (1 - e^{-2^k}) \\ &\geq c (L(p))^{-1/8}, \end{aligned}$$

where $c > 0$. We used Lemma 2.13 to get the second line, and the lower bound of Theorem 1.2 and (1.1) to get the third inequality. \square

3 Near-critical behavior: a fascinating self-organized near-criticality emerges

As promised, we start the discussion with the *near-critical regime* in standard percolation. Our goal here is to provide a self-contained explanation of the fact that the near-critical correlation length and the behavior of the percolation probability $\theta(p)$ as $p \searrow p_c$ are governed by the number of pivotals at criticality. Then we explain why this picture must be flawed in the case of random-cluster models.

3.1 Pivotal points govern the near-critical regime of percolation

Recall that the correlation length is defined as follows: fix some small $\epsilon > 0$. For any $n \geq 1$, let R_n be the $[0, n] \times [0, n]$ square for bond percolation on \mathbb{Z}^2 , and for any $p = p_c + \Delta p > p_c$, define

$$L_\epsilon(p) = L(p) := \inf \{n \geq 1 \text{ s.t. } \mathbb{P}_p(\mathcal{C}_v(R_n)) \geq 1 - \epsilon\},$$

where \mathbb{P}_p is the probability measure of bond-percolation with parameter p . Let us start by explaining the fact that things look supercritical above $L(p)$. For $n \geq L(p)$, the probability to have a left-to-right crossing in the box $[0, n]^2$ is larger than $1 - \epsilon$ by definition. Russo-Seymour-Welsh theory (see [Gri99, Lemma 11.73]) implies that the probability to have a left-to-right crossing in the rectangle $[0, 3n] \times [0, n]$ is greater than $1 - g(\epsilon)$, where $g(\epsilon)$ goes to 0 as $\epsilon \rightarrow 0$. Then, using well-known arguments, one can show that the “geometry” of ω_p above $L(p)$ stochastically dominates a certain **supercritical** percolation model of parameter $1 - \phi(g(\epsilon))$, where $\phi(x)$ goes to 0 as $x \rightarrow 0$. In this sense, things indeed look supercritical above $L(p)$. This step would have been simpler if one had worked directly with long rectangles $[0, 3n] \times [0, n]$ in the definition of $L(p)$ instead of the symmetric R_n . However, the symmetry of R_n will be relevant to the explanation of the following opposite fact: things look critical below $L(p)$. For this, we need to obtain RSW estimates for the dual percolation. By planar duality, if $n < L(p)$, then the probability to have a top-to-bottom **dual** crossing in R_n is greater than ϵ . Russo-Seymour-Welsh theory then implies

that the probability to have a dual left-to-right crossing in $[0, 3n] \times [0, n]$ is greater than $\psi(\epsilon)$ (where $\psi(\epsilon) \rightarrow 0$ when $\epsilon \rightarrow 0$). This means that at scales smaller than $L(p)$ we do have the uniform crossing probabilities that make the configuration look critical.

Goal: How can one estimate $L(p)$ as a function of $|p - p_c|$?

As explained in the Introduction, it is quite simple to guess what $L(p)$ should be. Indeed, the quantity is naturally related to thermodynamical quantities of the model via *Kesten's scaling relations*:

Theorem 3.1 ([Kes87], see also [Wer07, Nol08] for modern expositions, and [GPS] for an alternative approach). *For $L(p) = L_\epsilon(p)$, one has*

$$L(p)^2 \alpha_4(L(p)) \asymp \frac{1}{|p - p_c|}, \quad (3.1)$$

$$\theta(p) \asymp \alpha_1(L(p)), \quad (3.2)$$

where the constants in \asymp depend on $\epsilon > 0$.

Sketch of proof. Let us start with the first relation. In the following, let f_n denote the indicator function of the left-to-right crossing event in the domain R_n .

The intuition was already given in the Introduction: for crossing events of scale n (i.e., for rectangles of diameter $\asymp n$), there are $\Theta(n^2 \alpha_4(n))$ points on average which are pivotal for the crossing event. Now, in the standard monotone coupling, from ω_{p_c} to ω_p , each of these pivotal points flips with probability of order $|p - p_c|$. Therefore, it is tempting to believe that as far as $n^2 \alpha_4(n) |p - p_c| \ll 1$, it is unlikely for the crossing event to change, while once $n^2 \alpha_4(n) |p - p_c| \gg 1$, many pivotal points are flipped and things should start being highly connected.

A few things are a bit “fishy” in this intuition: one of them is that one might have $f_n(\omega_p) = 1$ and $f_n(\omega_{p_c}) = 0$ together with the fact that from ω_{p_c} to ω_p , none of the initial pivotal points for f_n had been switched: the pivotal switch might happen at a point that was not pivotal originally. On the way from ω_{p_c} to ω_p , if one stayed at equilibrium, as it is the case, for example, in dynamical percolation, one would still be able to conclude something based on such considerations, but one difficulty here is that as we follow the monotone coupling, we leave critical percolation.

The nice idea from [Kes87] to overcome this near-critical bias is to apply Russo's formula simultaneously to the crossing event as well as to the **four-arm** event. Indeed, for the crossing event, one can check that as long as $n \leq L(p) = L_\epsilon(p)$,

$$\begin{aligned} \frac{d}{dp} \mathbb{E}_p[f_n] &= \sum_{\text{sites } x} \mathbb{P}_p[x \text{ is pivotal}] \\ &\asymp n^2 \alpha_4^p(n). \end{aligned} \quad (3.3)$$

To go from the first line to the second one is in fact non-trivial: one needs to prove that below the correlation length, the main contribution in Russo's formula comes from bulk points rather than boundary points. The technology involved here is quasi-multiplicativity and separation of arms. One can prove these even if we are not at the critical point, since as far as $n \leq L(p)$, one still has RSW estimates both for the primal and the dual model.

Now the key observation is the following one:

$$\begin{aligned}
\left| \frac{d}{dp} \alpha_4^p(n) \right| &\leq \sum_x \mathbb{P}_p[x \text{ is pivotal for the 4-arm event}] \begin{cases} \text{using Russo's} \\ \text{formula again} \end{cases} \\
&\leq O(1) \alpha_4^p(n/3) \sum_{|x| \geq 2n/3} \mathbb{P}_p[x \text{ has the 4-arm event to distance } n/3] \\
&\leq O(1) \alpha_4^p(n) n^2 \alpha_4^p(n). \tag{3.4}
\end{aligned}$$

The second inequality uses quasi-multiplicativity along with a dyadic summation to show that the main contribution arises from large-scale pivotal points. The third inequality uses quasi-multiplicativity again.

The combination of (3.3) and (3.4) implies that for all $n < L(p)$, the variation of $p \mapsto \log(\alpha_4^p(n))$ is controlled (up to constants which depend only on ϵ) by the variation of $p \mapsto \mathbb{E}_p[f_n]$ which is of course bounded. This implies that

$$\alpha_4^p(n) \asymp \alpha_4(n), \tag{3.5}$$

where the constants involved in \asymp depend only on $\epsilon > 0$.

Now integrating (3.3) and using (3.5), one can conclude about the correlation length: for $n \leq L(p)$,

$$\begin{aligned}
\mathbb{E}_p[f_n] - \mathbb{E}_{p_c}[f_n] &\asymp \int_{p_c}^p n^2 \alpha_4^u(n) du \\
&\asymp |p - p_c| n^2 \alpha_4(n).
\end{aligned}$$

In particular, for $n = L(p)$, since $\mathbb{E}_p[f_n] - \mathbb{E}_{p_c}[f_n] \asymp 1$, we obtain our desired estimate (3.1). \square

The philosophy behind the previous argument is that, as far as few pivotal points are touched, percolation remains “critical”. This is what we called **near-critical stability** in Phenomenon 1.5. In particular, critical exponents remain unchanged below $L(p)$. We have seen this in (3.5) for the case of $\alpha_4(n)$, but the same argument works for the one-arm event $\alpha_1(n)$: namely, for $n < L(p)$,

$$\alpha_1^p(n) \asymp \alpha_1(n).$$

This immediately implies the second scaling relation:

$$\begin{aligned}
\theta(p) &:= \mathbb{P}_p[0 \leftrightarrow \infty] \\
&\asymp \mathbb{P}_p[0 \leftrightarrow \partial B(0, L(p))] \begin{cases} \text{since above } L(p), \text{ there} \\ \text{is a “dense” infinite cluster} \end{cases} \\
&= \alpha_1^p(L(p)) \\
&\asymp \alpha_1(L(p)). \tag{3.6}
\end{aligned}$$

The knowledge of critical arm-exponents $\xi_1(1)$ and $\xi_4(1)$ for site-percolation on the triangular lattice \mathbb{T} (see [SW01, LSW02]) allows for an estimation of $L(p)$ and $\theta(p)$:

$$\begin{aligned}
L_{\mathbb{T}}(p) &= (p - p_c)^{-4/3+o(1)} \\
\theta_{\mathbb{T}}(p) &= (p - p_c)^{5/36+o(1)}
\end{aligned}$$

as $p \searrow p_c = 1/2$.

As we discussed at length in the Introduction, the first scaling relation (3.1) does not hold in the FK-Ising case (see (1.4) and Phenomenon 1.6). Nevertheless, note that the heuristics of the second scaling relation should work in great generality. Contrary to the first scaling relation, it gives the right prediction for the correlation length. Indeed, the thermodynamical quantities $L(p)$ and $\theta(p)$ have their analogues in the FK-Ising case. Onsager's determination of the magnetization, together with the Edwards-Sokal coupling implies that

$$\theta_{\mathbb{Z}^2}^{\text{FK}}(p) := \phi_{p,2}(0 \leftrightarrow \infty) \asymp |p - p_c|^{1/8} \quad (3.6)$$

and

$$\phi_{p_c,2}(0 \leftrightarrow \partial[-n, n]^2) \asymp n^{-1/8}. \quad (3.7)$$

From these two relations, the second scaling relation (which does not harness any pivotal event) implies that the correlation length should behave as $1/|p - p_c|$ for FK-Ising, which is the right prediction. Also note that (3.7) has been proved using conformal invariance techniques in [CHI12]. It would be interesting to make sense of the second scaling relation in the FK-Ising case in order to provide a derivation of the exponent $1/8$ for the magnetization which would be independent of Onsager's computation. Half of this is achieved by Theorem 1.4.

Let us conclude this paragraph by mentioning another result proved in [Kes87]. The correlation length is sometimes defined as the inverse rate of exponential decay of the connectivity probabilities in the subcritical regime:

$$\frac{1}{L(p)} := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}_p[(0, 0) \leftrightarrow (n, 0)].$$

Kesten proved that this definition is consistent with the definition in terms of crossing probabilities. Nonetheless, this correspondence is known only in the percolation case and is not fully understood in the FK-Ising case. Furthermore, this definition is less tractable when studying near-critical regimes. It is therefore useless to consider existing estimates on the inverse rate of exponential decay for the FK-Ising model (see [BDC10b] *e.g.*) to prove that the correlation length defined as before behaves as $|p - p_c|^{-1}$.

3.2 Specific heat of the random-cluster model

We would now like to understand the behavior of random-cluster models when p varies from 0 to 1. The first non-trivial effect which occurs in the near-critical random-cluster model is the fact that the derivative of the edge-intensity blows up around p_c for $q \geq 2$. This implies that edges appear much faster in any monotone coupling near p_c than they do for percolation. Define the edge-intensity function as follows: for all $p \in [0, 1]$, let

$$\mathcal{I}(p) := \phi_{p,q}(e \text{ is open}),$$

where e is any edge of \mathbb{Z}^2 . It is not hard to check that at the critical (and self-dual) point $p_c(q) = \frac{\sqrt{q}}{1+\sqrt{q}}$, one has

$$\mathcal{I}(p_c) = \frac{1}{2}.$$

A quantity relevant to us is the derivative in p of the edge-density $d\mathcal{I}(p)/dp$. It corresponds to the average rate at which new edges appear in any possible monotone coupling

$(\omega_{p,q})_{p \in [0,1]}$. For integer $q \geq 2$, this quantity turns out to be linked to the so-called **specific heat** of the q -Potts model, defined by

$$C^{\text{Potts}(q)}(\beta) := -\beta^2 \frac{\partial}{\partial \beta} \mathbb{E}_\beta[H] = \beta^2 \text{Var}_\beta[H], \quad (3.8)$$

where H is the total energy of the system, and where the second identity can be easily proved by manipulating the partition function $Z_\beta^{\text{Potts}(q)}$ (recall here, e.g., that $\mathbb{E}_\beta[H] = -\partial_\beta \log Z_\beta^{\text{Potts}(q)}$). By the Edwards-Sokal coupling, the partition function $Z_\beta^{\text{Potts}(q)}$ equals the partition function $Z_{p,q}$ of the FK(p, q) model with $e^{-2\beta} = 1 - p$, hence the specific heat can also be defined for the FK(p, q) model, to be denoted by $C^{\text{FK}_q}(p)$, using the second derivative of the free energy $\log Z_{p,q}$. This works now for any $q \geq 0$, except that the $q = 1$ percolation case (where $Z_{p,1} \equiv 1$) needs a limiting procedure $q \rightarrow 1$, which yields the second derivative of the expected number of clusters per site, or $d^2 \mathbb{E}_p[1/|\mathcal{C}_x|]/dp^2$, where \mathcal{C}_x is the cluster of any fixed site x .

The relationship between the derivative of the edge-intensity and the specific heat per site is detailed in [GH], but the point is that for $q \geq 2$, they are within bounded factors from each other. So, for $q = 2$, from the results on the specific heat of the Ising model known since [Ons44, FF69], one obtains in the infinite volume case that, at $p = p_c(2)$,

$$\frac{d}{dp} \Big|_{p=p_c} \mathcal{I}(p) = \infty.$$

More precisely, one can extract the following logarithmic behavior around p_c :

$$\frac{d}{dp} \mathcal{I}(p) \sim a \log \frac{1}{|p - p_c|},$$

as $p \rightarrow p_c$. The finite-volume study of the specific heat ([Ons44, FF69]) leads to the following estimate: let \mathbb{T}_n be the torus $\mathbb{Z}^2/n\mathbb{Z}^2$ and let $p \mapsto \mathcal{I}_n(p)$ denote the edge-intensity for random-cluster model on \mathbb{T}_n , then

$$\frac{d}{dp} \Big|_{p=p_c} \mathcal{I}_n(p) \asymp \log n.$$

The extension to planar domains $\Omega_n := \frac{1}{n}\mathbb{Z}^2 \cap \Omega$ will be carried out in [GH], based on the recent results from [Hon10] and [BdT08, BdT09].

In conclusion, these estimates show that in a window of size n , as one raises p near p_c , more edges will suddenly arrive. Nevertheless, the discrepancy is only logarithmic, hence it is not sufficient to explain the error (1.4) in the power law of the correlation length.

3.3 The monotone increasing Markov process on cluster configurations

Let us present a monotone coupling μ between random-cluster models with fixed $q \geq 1$, which was first considered by Grimmett in [Gri95]. It turns out that it is non-trivial to construct explicitly such a measure μ (note that in contrast the existence of abstract monotone couplings follows easily from a generalized Strassen's theorem). Instead of constructing explicitly a coupling μ , Grimmett obtains it for a graph $G = (V, E)$ as the invariant measure μ of a natural Markov process on the space $\Omega := [0, 1]^E$. This technique is usually employed to simulate a single Gibbs measure and we propose to provide this example first.

Heat-bath dynamics. Assume we wish to simulate the random-cluster measure $\phi_{G,p,q}^0$ on the graph $G = (V, E)$. First note that the random-cluster model with parameters (p, q) has the following property:

Property 3.2. *For any edge $e = [xy]$,*

$$\phi_{G,p,q}^0(e \text{ is open} \mid \omega \text{ on } G \setminus \{e\}) = \begin{cases} p & , \text{ if } x \overset{\omega}{\longleftrightarrow} y \text{ in } G \setminus \{e\} \\ \frac{p}{p+(1-p)q} & , \text{ otherwise.} \end{cases}$$

This “almost local” rule for the conditional law of an edge e knowing the external environment enables us to consider the **heat-bath dynamics**:

Definition 3.3 (Heat-bath dynamics or Sweeny algorithm). *Let $G = (V, E)$ be any finite graph. The random-cluster heat-bath dynamics on G is defined as follows: each edge $e \in E$ is updated at rate 1 (the exponential clocks being independent) and when a clock rings at e , its status $\omega(e)$ is resampled according to the conditional law given in Property 3.2.*

It is straightforward to check that this dynamics has the following properties: it is a reversible Markov chain with state space $\{0, 1\}^E$ and its invariant measure is $\phi_{G,p,q}^0$. This dynamics has been studied both for theoretical reasons (see [Gri95, Gri06]) and practical ones (see [DGS07] for a good account of recent works). For instance, via the Edwards-Sokal coupling, it turns out to provide a faster way than classical Glauber dynamics to sample Ising models (at least in dimension $d = 2$). More sophisticated algorithms are known for integer values of q (for example, the Swendsen-Wang algorithm). However, the above dynamics has the advantage that it works for all real values of q and is probably more tractable for rigorous analysis.

Grimmett’s dynamics and monotone coupling. We now describe briefly Grimmett’s monotone coupling (see [Gri95, Gri06] for a detailed exposition). Let $G = (V, E)$ be a finite subgraph of \mathbb{Z}^2 and Ω be the space $[0, 1]^E$. Each $Z \in \Omega$ decomposes into a monotone family of edge configurations $(\omega_p)_{0 \leq p \leq 1} = (\omega_p(Z))_{0 \leq p \leq 1}$, where for each $p \in [0, 1]$ and any $e \in E$:

$$\omega_p(Z)(e) := 1_{Z(e) \leq p}.$$

The goal is to find a measure $\mu = \mu_G$ on Ω in such a way that all the “projections” $\omega_p(Z)$ with $Z \sim \mu$ follow the random-cluster probability measure of parameters (p, q) on $\{0, 1\}^E$ with free boundary conditions. It is not hard to see what this Markov process $(Z_t)_{t \geq 0}$ should be. Assuming that for all $p \in [0, 1]$, the projection $\omega_p(Z_t)$ is also a Markov process and that its invariant measure is $\phi_{G,p,q}^0$, then it is natural to expect $\omega_p(Z_t)$ to be the heat-bath dynamics that we defined previously. Indeed, if $(Z_t)_{t \geq 0}$ is such that each edge is updated at rate one (the exponential clocks on each edge being independent), this means that **simultaneously** for all $p \in [0, 1]$, the law of the update at e needs to be compatible with Property 3.2. For any $e = \langle x, y \rangle \in E$, let $\mathcal{D}_e \subset \{0, 1\}^E$ be the event that there is a path of open edges in $E \setminus \{e\}$ connecting x and y . For any $e \in E$ and any $Z \in \Omega$, define

$$T_e(Z) := \inf\{p \in [0, 1] \text{ s.t. } \omega_p(Z) \in \mathcal{D}_e\}.$$

Assume one is running the dynamics and that at a time t the edge e rings. Let Z_{t-} be the current configuration (before the update). Let \mathcal{U}_e be the random variable corresponding to the new label at e knowing Z_{t-} . In particular, we know the value of $T = T_e(Z_{t-})$. Since $p \geq T$ is equivalent to $\omega_p(Z_{t-}) \in \mathcal{D}_e$, in order for Z_t to match the heat-bath dynamics on

the projection $\omega_p(Z_t)$ for all p simultaneously, conditionally on the value $T = T_e(Z_{t-})$, the update variable \mathcal{U}_e must satisfy

$$\mathbb{P}[\mathcal{U}_e \leq p] := \begin{cases} p & \text{if } p \geq T \\ \frac{p}{p+(1-p)q} & \text{if } p < T. \end{cases} \quad (3.9)$$

Fortunately, $q \geq 1$ implies that this is a valid distribution function, hence we can simply define \mathcal{U}_e to be a sample from this distribution. Note that \mathcal{U}_e has an absolutely continuous part plus a **Dirac point mass** (for $q > 1$) on T , namely $[T - \frac{T}{T+(1-T)q}] \delta_T$.

This discussion motivates the introduction of the Markov chain Z_t on the state space $X_\Lambda = [0, 1]^{E(\Lambda)}$ where labels on the edges are updated at rate one according to the above conditional law (given by \mathcal{U}_e , $e \in E(\Lambda)$). This is precisely the dynamics that was considered by Grimmett in [Gri95] (see also [HJL02] where this dynamics was revisited).

Constructing an infinite-volume version of the previous dynamics is not straightforward. Nevertheless, one has the following asymptotic statement from [Gri95, Gri06].

Proposition 3.4 (Infinite Volume Limit [Gri95]). *For each $n \geq 1$, let $\Lambda_n := [-n, n]^d$. Let ξ be some initial configuration in $X := [0, 1]^{E(\mathbb{Z}^2)}$. Consider the random-cluster heat-bath dynamics $Z_t^{\Lambda_n}$ on Λ_n with **free** boundary conditions and which starts from the initial state $Z_0^{\Lambda_n} \equiv \xi|_{\Lambda_n}$. Then, as $n \rightarrow \infty$, the process $(Z_t^{\Lambda_n})$ weakly converges to a **Markov** process $(Z_t^{\text{free}})_{t \geq 0}$ which starts from the initial configuration $Z_0^{\text{free}} = \xi$.*

Furthermore, as $t \rightarrow \infty$, Z_t^{free} weakly converges to an invariant measure μ on X .

*If, in the limiting procedure, one uses **wired** boundary conditions instead, one obtains at the limit a Markov process $(Z_t^{\text{wired}})_{t \geq 0}$. The processes Z_t^{wired} and Z_t^{free} might possibly have different transition kernels but they both have the same μ as the **unique** invariant measure.*

The underlying dynamics here is **non-Fellerian**, and the limiting Markov process in the above theorem is derived from the monotonicity properties inherent to the dynamics. In particular, the relationship between this Markov process and its formal generator (we will not write it down explicitly here) would need to be investigated. This seems to be a non-trivial task for the present dynamics. Therefore, we will not assume any explicit **transition rule** for the infinite-volume dynamics Z_t^{free} (or Z_t^{wired}) and will restrict ourselves to the “compact case”.

The Markov property of the monotone coupling. Let us prove that Grimmett’s coupling leads to a monotone increasing **Markov** process (as p varies) on the cluster configurations, *i.e.*, on the space $\{0, 1\}^E$. We will not rely on this Markovian property later on. Nevertheless, it provides a nice picture of the self-organization scheme near $p_c(q)$. Namely, as one raises p near p_c , new edges arrive in a complicated fashion yet depending only on the current configuration ω_p .

Proposition 3.5. *Let $G = (V, E)$ be a finite subgraph of \mathbb{Z}^2 . Let Z be sampled according to the law μ . Then the monotone family of projections $(\omega_p(Z))_{0 \leq p \leq 1}$, seen as a random process in the “time” variable p , is a non-decreasing **inhomogeneous Markov process** on the space $\{0, 1\}^E$.*

Proof. We wish to prove that conditioned on the projections $(\omega_u(Z))_{0 \leq u \leq p}$, the conditional law of the higher configurations $(\omega_u(Z))_{p \leq u \leq 1}$ depends only on $\omega_p(Z)$. To achieve this, it is

enough to prove Lemma 3.6 below. Before stating the lemma, we introduce some notation. For $p \in [0, 1]$, decompose the configuration Z into the triple $(\omega_p, Z^{\leq p}, Z^{> p})$ defined as

$$\omega_p = \omega_p(Z_\Lambda); \quad Z^{\leq p} = \begin{cases} Z & \text{if } Z \leq p \\ 1 & \text{otherwise} \end{cases}; \quad Z^{> p} = \begin{cases} Z & \text{if } Z > p \\ 0 & \text{otherwise} \end{cases}.$$

Note that

$$\omega_p = \omega_p(Z^{\leq p}) = \omega_p(Z^{> p}), \quad (3.10)$$

and that Z can be recovered from the triple $(\omega_p, Z^{\leq p}, Z^{> p})$.

Lemma 3.6. *Conditioned on the value of the first component ω_p , the two other components $Z^{\leq p}$ and $Z^{> p}$ are conditionally independent.*

Proof of the lemma. Fix $p \in [0, 1]$ and omit it from the notation $\omega = \omega_p$ to make space for a time variable t .

We basically follow the construction of the measure μ as the limiting measure of the Markov process Z_t , except that we divide the randomness used along the Markov chain into three components, the second and third being independent conditionally on the first one. Namely, define a Markov process

$$(\omega_t, Z_t^{\leq p}, Z_t^{> p})_{t \geq 0} \in \{0, 1\}^{E(\Lambda)} \times [0, 1]^{E(\Lambda)} \times [0, 1]^{E(\Lambda)},$$

where edges are updated at rate one, in such a way that the relations (3.10) between the three coordinates hold for all $t \geq 0$. To be consistent at $t = 0$, the process starts either from the empty state $(\omega_0, Z_0^{\leq p}, Z_0^{> p}) \equiv (\mathbf{0}, \mathbf{1}, \mathbf{1})$ or the full state $(\mathbf{1}, \mathbf{0}, \mathbf{0})$, where $\mathbf{0}$ and $\mathbf{1}$ denote the vectors all 0 and all 1 respectively. Then, instead of sampling \mathcal{U}_e directly, let us proceed stepwise: first look whether ω_{t-} satisfies \mathcal{D}_e or not. If it does, then let $\omega_t(e) := 1$ with probability p . If $\omega_{t-} \notin \mathcal{D}_e$, then let $\omega_t(e) := 1$ with probability $p/(p + (1-p)q)$. This is exactly the heat-bath dynamics for $\phi_{G,p,q}^0$. Note that this part of the dynamics does not use at the two components $(Z^{\leq p}, Z^{> p})$.

Let us describe how to update the component $Z_t^{\leq p}$. If, after the update, $\omega_t(e)$ equals 0, then we fix $Z_t^{\leq p}(e) := 1$. Otherwise (if $\omega_t(e) = 1$), we use the following variable:

$$T_e^{\leq p}(Z^{\leq p}) := \inf\{u \in [0, p] : \omega^u(Z^{\leq p}) \in \mathcal{D}_e\}.$$

Note that $T_e^{\leq p}(Z^{\leq p}) = T_e(Z)$ on the event $T_e(Z) \leq p$. Otherwise (i.e. $\omega_p(Z) \notin \mathcal{D}_e$), we set $T_e^{\leq p} = p$. In either case, it is important here that no information about the third component $Z^{> p}$ has been used.

Next, recall the update random variable \mathcal{U}_e from the previous subsection (see (3.9)). It needed as an input the value of $T_e(Z_{t-})$. Let $\mathcal{U}_e^{\leq p}$ be the same random variable here, with input the value of $T_e^{\leq p}(Z_{t-}^{\leq p})$. Remembering that we are in the case $\omega_t(e) = 1$, update the value of $Z_t^{\leq p}$ as follows, independently of everything:

$$Z_t^{\leq p}(e) \sim \mathcal{L}[\mathcal{U}_e^{\leq p} \mid \mathcal{U}_e^{\leq p} \leq p],$$

where \mathcal{L} stands for the law of the variable. We define $(Z_t^{> p})$ in the same fashion, using $\mathcal{U}_e^{> p}$. In particular, the evolutions of $(Z_t^{\leq p})$ and $(Z_t^{> p})$ are sampled out of the evolution of (ω_t) plus some randomness in each case that are independent of each other, hence the conditional independence of $Z^{\leq p}$ and $Z^{> p}$ is satisfied.

To conclude the proof, one just has to notice that if one defines

$$Z_t := \begin{cases} Z_t^{\leq p} & \text{if } \omega_t(e) = 1 \\ Z_t^{>p} & \text{else,} \end{cases}$$

then $(Z_t)_{t \geq 0}$ is exactly the Markov chain which was considered by Grimmett in [Gri95]. (This is not hard to check; an important feature here is that if $T_e > p$, then the conditional law $\mathcal{L}[\mathcal{U}_e \mid \mathcal{U}_e \leq p]$ does not depend on the exact value of T_e , and a similar thing holds for $\mathcal{U}_e^{>p}$ when $T_e \leq p$.) In particular, from [Gri95], it converges to the unique invariant measure μ_Λ , which inherits its conditional independence property. This finishes the proof of Lemma 3.6 and hence of Proposition 3.5. \square

Proposition 3.7. *This Markovian property extends to the infinite volume limit μ on $X = [0, 1]^{E(\mathbb{Z}^2)}$.*

Indeed, the same procedure works, but one has to be a bit careful with the initial state of our Markov chain: following [Gri95], with the slightly asymmetric projection convention we have chosen, we need to start from the full state $(\omega_0, Z_0^{\leq p}, Z_0^{>p}) \equiv (\mathbf{1}, \mathbf{0}, \mathbf{0})$. In the case $q = 2$, this is not very important since there is a unique infinite volume limit for all values of p . However for larger values of q , these considerations do matter. We will not enter in more detail here; see [Gri95, Gri06] for a detailed exposition on the infinite volume limit of Z together with its projection $\omega_p(Z)$.

3.4 Existence of emerging clouds when $q = 2$

We now give a concrete manifestation in Grimmett's coupling of the self-organized behavior appearing in any monotone coupling of random-cluster models, explained in Phenomenon 1.6. We restrict ourselves to the finite case, since the transition rule for the infinite volume Markov process $(Z_t)_{t \geq 0}$ has not been established. Let then Λ be a finite box in \mathbb{Z}^2 (or a torus $\mathbb{Z}^2/n\mathbb{Z}^2$).

To start with, let us define properly the notion of **cloud** for a finite graph $\Lambda = (V, E)$. Given a sample $Z = Z_\Lambda \in [0, 1]^E$ from Grimmett's monotone coupling μ_Λ , for an edge $e \in E$, let $\text{cloud}(e)$ be the set of edges which appear simultaneously with e :

$$\text{cloud}(e) := \{f \in E \text{ s.t. } Z(f) = Z(e)\}.$$

The following proposition gives the first hint of some “non-linear” behavior:

Proposition 3.8. *Fix $q > 1$. For any $N \geq 1$, let $(\omega_p(Z_\Lambda))_{p \in [0, 1]}$ be a monotone coupling in the box Λ . The probability that clouds of at least N edges appear simultaneously in $\omega_p(Z_\Lambda)$ at some $p \in (0, 1)$ converges to 1 when the size of the box $\Lambda \nearrow \mathbb{Z}^2$.*

This proposition is very easy to prove, yet one already sees here that the monotone Markovian coupling $(\omega_{\Lambda, p})_{0 \leq p \leq 1}$ has a nature that is very different from the $q = 1$ case.

Proof. Let us consider the sets E_1 , E_2 and E_3 in Λ (which is assumed to be large enough) as defined in Fig. 3.1.

Now let us sample $Z_0 = Z_{t=0}^\Lambda$ according to the invariant measure μ_Λ , and let us run the dynamics for a unit time. With positive probability, all edges in E are updated and their labels at time 1 satisfy the following: all labels in E_2 are smaller than $1/4$, the edge $e_0 = \langle (0, 0), (1, 0) \rangle$ gets a label in $(1/4, 1/2)$, and all other labels in $E_1 \cup E_3$ are larger than

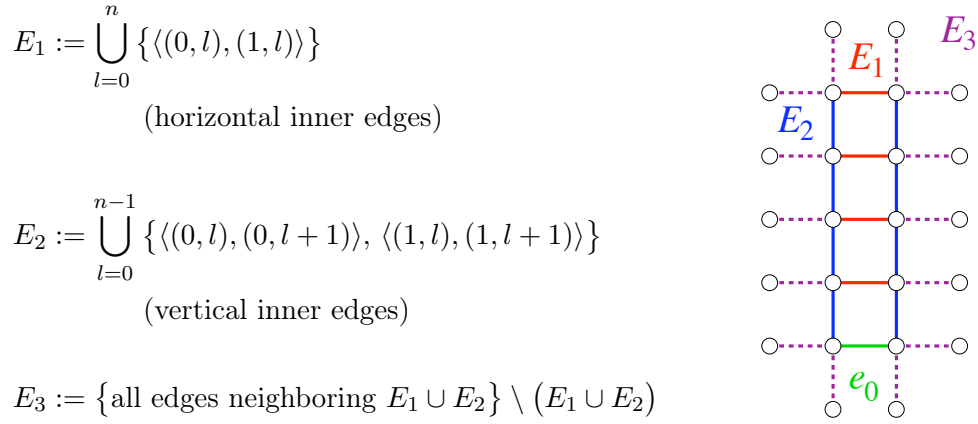


Figure 3.1: The definition of the sets E_1 , E_2 , E_3 and the edge $e_0 \in E_1$.

3/4. Under such circumstances, all edges $e \in E_1 \setminus \{e_0\}$ are such that $T_e(Z_{t=1}) = Z_1(e_0)$. It could be that this situation evolves later on, but we have that, with positive probability, none of the edges in $E_2 \cup E_3 \cup \{e_0\}$ are updated from time 1 to time 2. Knowing this, again with positive probability, all edges in E_1 are updated from time 1 to time 2 and all of them take exactly the value $u := Z_1(e_0)$ (this is due to the Dirac mass δ_u in the law \mathcal{U}_e). Since we started at equilibrium, $Z_{t=2}$ has the equilibrium law, and edges in E_2 are all open or all closed in the projections of $Z_{t=2}$. This shows that with positive probability, at least N edges appear simultaneously as one raises p .

Now, if Λ is getting very large, we can divide the box into a lattice of $2N \times 2N$ squares. Starting from $Z_{\Lambda,0} \sim \mu_\Lambda$, the above strategy works in each box independently of what happens in other boxes. Stated like that, it looks wrong, since obviously the dynamics itself is not independent from one square to another, but all that is needed in the above procedure is a positive lower bound on the probability that this “scenario” happens. Using the structure of the dynamics, it is not hard to see that if y_1, \dots, y_K denote the indicator functions of the events that the scenario happened in the squares $i \in \{1, \dots, K\}$, then there is an independent product of Bernoulli $\epsilon > 0$ variables which is stochastically dominated by our vector (y_1, \dots, y_K) . In particular, the emergence of clouds is somewhat ergodic in the plane.

By changing slightly the argument, one can show that there are such clouds for any open interval of the variable $p \in [0, 1]$. \square

The above proof is not quantitative at all. Therefore, many natural questions on these **emerging clouds** remain: how do they look, how large are they, how does their law depend on the level p at which they appear? In particular, recovering the correlation length from such geometric considerations appears to be quite a challenging program. We shall discuss some of these questions in the next subsection.

An intuitive explanation for the clouds. We end this subsection by a hand-waving argument why these clouds of simultaneously opening edges appear and may play an important role in the dynamics of any monotone coupling. Consider a monotone coupling $(\omega_p, \omega_{p+\Delta p})$. Due to the factor $q^{\# \text{ clusters}}$ in the partition function, FK configurations ω_p tend to have as many clusters as possible. Without this factor, one would be in the case of $q = 1$, *i.e.*, standard percolation, and the edge intensity would be exactly p . With $q > 1$,

the random-cluster configuration tries to maximize the number of clusters, hence the edge-intensity drops to a smaller value $\mathcal{I}(p) < p$. In some sense, there is a fight between entropy (under the product measure $p^{\#\text{ open edges}}(1-p)^{\#\text{ closed edges}}$, most configurations have edge-intensity p) and energy (which would correspond here to $-\log(q^{\#\text{ clusters}})$). When one goes from p to $p + \Delta p$, new edges are added due to the entropy effect, but in such a way that not so many clusters will merge into a single one. A good strategy for adding many edges without a significant increase in energy is the following **storing mechanism**. Say we have two “neighboring” large clusters in ω_p with closed edges going from one to the other (these closed edges are then large-scale pivotal edges). Once we decide to open one of them, it does not cost more energy to open a few others.

Now, we have just seen that the monotone coupling is Markovian in p : in particular, the only way for this storing mechanism to actually happen is to have some values of p where the system can simultaneously open several edges. This indeed can happen, due to the atom in the update distribution, as shown in Lemma 3.8, and the construction there was indeed a simple example of edges arriving simultaneously between two neighboring large clusters (the two components of E_2).

It is worth noticing that this heuristic explanation (based on entropy/energy considerations plus the Markov property) hints that this “non-linear phenomenon” should be much stronger near the critical point. Indeed, near $p_c(q)$, there are many neighboring large clusters (i.e., many large scale pivotal points), which makes the storing mechanism more efficient. Away from criticality, this is not the case anymore. This intuition explains, for example, why we observe a blow-up of the derivative of the edge-intensity near p_c , and why the emerging clouds are more important there.

3.5 Questions on the structure of emerging clouds

Finite volume case. The previous subsection shows that there are non-trivial clouds with positive probability. Let us consider the case of $G = \Lambda_n := [-n, n]^2$ with free boundary conditions. (Another natural choice would be to consider discrete tori $\Lambda_n := \mathbb{Z}^2/n\mathbb{Z}^2$). We strongly suspect the following behavior:

Question 1 (Macroscopic clouds near p_c). *For all $n \geq 1$, with μ_{Λ_n} -probability at least a universal constant $c > 0$, there is at least one **macroscopic cloud** in Λ_n , i.e., whose diameter is larger than cn . Furthermore, with probability going to 1 as $n \rightarrow \infty$, the labels of such macroscopic clouds concentrate around the critical value $p_c(q = 2)$.*

To answer such a question, it is natural to run the dynamics at equilibrium (i.e., $Z_0^n \sim \mu_{\Lambda_n}$) for a short amount of time that is given precisely by the rescaling

$$\tau_n := \frac{1}{n^2 \xi_4(2)} = n^{-13/24 + o(1)}.$$

Doing so, only finitely many macroscopic pivotal edges will be resampled, and it is easy to convince ourselves that with positive probability at least two of them will pick the same label thus creating a macroscopic cloud. This intuition is close to being rigorous, since we have at our disposal a ‘stability property’ from the forthcoming [GP] which suggests that the “geometry” of $Z_{t=\tau_n}^n$ could be recovered with high precision from Z_0^n plus the updates of the initially macroscopically important edges (neglecting the “smaller” updates). However, the stability result holds only for ω_{p_c} , not the entire coupling Z . Thus, a certain control on the *concentration* of the labels around p_c would be helpful for both parts of Question 1.

The intuition that big clouds should appear only around the critical point can be translated into the following conjecture:

Question 2 (Local clouds away from p_c). *For any $\delta > 0$, emerging clouds with labels outside of $(p_c - \delta, p_c + \delta)$ are **local** in the sense that the largest such cloud in Λ_n should be of logarithmic size.*

A natural way to attack this question would be via a **coupling** argument. Namely, construct a coupling $(Z_{\Lambda_n}^{\geq p_c + \delta}, \tilde{Z}_{\Lambda_n}^{\geq p_c + \delta})$ (see the notation in Subsection 3.3) whose marginals are $\mu_{\Lambda_n}^{\geq p_c + \delta}$, and whose coordinates are identical on a small neighborhood of the origin, but with probability at least λ^k (with $\lambda \in (0, 1)$), are independent of each other outside a box of size k (an exponential decay of correlations). Such a statement is proved for the supercritical (or subcritical) random-cluster measure $\phi_{\mathbb{Z}^2, p, q}$, yet the lack of a DLR (spatial Markov) property for our monotone coupling μ_{Λ_n} prevents us from extending the result to the coupling in an obvious way.

Finally, it would be interesting to prove quantitative results on the size of emerging clouds in the finite volume case (Λ_n) . This question is further discussed in [GH].

Infinite volume case. Clouds are well-defined objects: since there is a unique limiting measure $\mu_{\mathbb{Z}^2}$ of Grimmett's monotone coupling and for any $e \in E(\mathbb{Z}^2)$, $\text{cloud}(e)$ can still be defined relatively to a sample Z from $\mu_{\mathbb{Z}^2}$.

Question 3. *Prove that a.s. there exist non-trivial emerging clouds.*

This does not follow directly from the existence of non-trivial clouds in the finite volume case. Assuming the above question, the next natural question would be the following:

Question 4. *Is it the case that emerging clouds are a.s. finite when $q \leq 4$?*

Note that for large q , an infinite number of edges appear at $p_c(q)$. Again, [GH] contains some relevant discussion.

3.6 What about the influence of an edge?

To conclude this section, let us mention another natural approach to a “geometric” understanding of the near-critical random-cluster model.

As a continuation of the work by Kesten on near-critical percolation [Kes87], Russo's formula should be replaced by a slightly different formula. Fix an increasing event A . As in the case of percolation, the intuition suggests that the derivative of $\phi_{G, p, q}^\xi(A)$ with respect to p is mostly governed by the influence of one single edge, switching from closed to open. The following definition is therefore natural in this setting. The *(conditional) influence* on A of the edge $e \in E$, denoted by $I_A^p(e)$, is defined as

$$I_A^p(e) := \phi_{G, p, q}^\xi(A | e \text{ is open}) - \phi_{G, p, q}^\xi(A | e \text{ is closed}).$$

With this notation, we have the following formula:

Proposition 3.9 (See [Gri06]). *Let $q \geq 1$ and $\varepsilon > 0$; for any random-cluster measure $\phi_{G, p, q}^\xi$ with $p \in [\varepsilon, 1 - \varepsilon]$ and any increasing event A ,*

$$\frac{d}{dp} \phi_{G, p, q}^\xi(A) \asymp \sum_{e \in E} I_A^p(e),$$

where the constants in \asymp depend on q and ε only.

It is tempting to use this extension of Russo's formula to see what our results on the correlation length (Theorem 1.2) may imply on the *influences* $I_A^p(e)$. To avoid boundary issues, let us consider the case of the torus $\mathbb{T}_n := \mathbb{Z}^2/n\mathbb{Z}^2$, and let A_n be the event that there is an open circuit with non-trivial homotopy in \mathbb{T}_n . It is easy to check (by self-duality) that $\phi_{p_c,2}(A_n) \leq 1/2$. The results from Section 2 can easily be generalized to the torus. In particular, there exists a constant $\lambda > 0$ such that if $p_n := p_c(2) + \lambda \frac{\sqrt{\log n}}{n}$, then

$$\phi_{p_n,2}(A_n) \geq 3/4.$$

Using the above Proposition 3.9, this says that

$$\int_{p_c}^{p_c + \lambda \frac{\sqrt{\log n}}{n}} (I_{A_n}^p(e_{hor}) + I_{A_n}^p(e_{ver})) dp \geq \Omega(1) \frac{1}{n^2},$$

where e_{hor} and e_{ver} are any horizontal and vertical edges in \mathbb{T}_n . Since it is natural to expect that on the interval $[p_c, p_c + \lambda \frac{\sqrt{\log n}}{n}]$, *influences* behave reasonably smoothly, the following conjecture should hold.

Conjecture 3.10. *For any $n \geq 1, \lambda > 0, p \in [p_c - \lambda \frac{\sqrt{\log n}}{n}, p_c + \lambda \frac{\sqrt{\log n}}{n}]$ and any $e \in \mathbb{T}_n$,*

$$I_{A_n}^p(e) \geq c \frac{1}{n\sqrt{\log n}},$$

where $c = c(\lambda)$ is some positive constant.

In fact since it is reasonable to conjecture that in Theorem 1.2, one has actually $L_{\rho,\varepsilon}^\xi(p) \asymp |p - p_c|^{-1}$, one may strengthen the previous conjecture into the following one:

Conjecture 3.11. *For any $n \geq 1, \lambda > 0, p \in [p_c - \frac{\lambda}{n}, p_c + \frac{\lambda}{n}]$ and any $e \in \mathbb{T}_n$,*

$$c \frac{1}{n} < I_{A_n}^p(e) < c^{-1} \frac{1}{n},$$

where $c = c(\lambda)$ is some positive constant.

These conjectures are beyond reach with the techniques of the present paper.

4 Other values of q

As promised in the Introduction, most of this section will rely on predictions from physics to investigate what happens for general $q \in [1, 4]$.

4.1 Some useful critical exponents

Let us start by collecting several useful exponents.

- $\xi_1(q)$ denotes the *one-arm* exponent
- $\xi_4(q)$ denotes the *four-arm* exponent
- $\alpha = \alpha(q)$ describes the behavior of the **specific heat** near $p_c(q)$. That is, $C^{\text{FK}_q}(p) \approx |p - p_c(q)|^{\alpha(q)}$.

- $\beta = \beta(q)$ describes the behavior of the “magnetization”: this can be interpreted as

$$\phi_{p,q}(0 \leftrightarrow \infty) \approx (p - p_c(q))^{\beta(q)} \quad (\text{as } p \searrow p_c(q)).$$

- $\nu = \nu(q)$ corresponds to the correlation length: $L^{\text{FK}_q}(p) \approx |p - p_c(q)|^{-\nu(q)}$.
- $\eta = \eta(q)$ corresponds to the correlation function $\mathbb{P}_{p_c(q)}[x \leftrightarrow y] \approx |x - y|^{-\eta(q)}$. In particular, assuming RSW, this exponent is twice the *one-arm* exponent $\xi_1(q)$.

Let us summarize the physics predictions on these exponents in the following table. The expressions are simplified using the term $u = u(q) = \frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2}) = 2 - \frac{8}{\kappa(q)}$.

Exponents	predictions
$u = u(q)$	$\frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2})$
$\alpha = \alpha(q)$	$\frac{2(1-2u)}{3(1-u)}$
$\beta = \beta(q)$	$\frac{1+u}{12}$
$\nu = \nu(q)$	$\frac{2-u}{3(1-u)}$
$\eta = \eta(q)$	$\frac{1-u^2}{2(2-u)}$
$\xi_1 = \xi_1(q)$	$\frac{1-u^2}{4(2-u)}$
$\xi_4 = \xi_4(q)$	$\frac{5}{2} - \frac{3}{4}u - \frac{1}{2-u}$

Most of the previous critical exponents can be found, for example, in [Wu82], except the *four-arm* or *pivotal* exponent which is more of a geometric nature. This latter exponent is computed in [DCG] using SLE_κ calculations and assuming the (conjectured) correspondence

$$\kappa = \kappa(q) := \frac{4\pi}{\arccos(-\frac{\sqrt{q}}{2})}.$$

This SLE exponent was also derived by Wendelin Werner [Wer09a]. Assuming a proof of **conformal invariance** for the critical random-cluster model with parameter q as well as a proof of **quasi-multiplicativity** on the discrete level, this would say that

$$\alpha_4^{\text{FK}_q}(n) = n^{-\xi_4(q)+o(1)}.$$

Such an estimate is of course far from reach at the moment, except for $q = 1$ (see [SW01]) and $q = 2$ (see [DCG]), but in this section we will assume that it holds.

These exponents are not all independent of each other: there are scaling and hyperscaling relations between them that are expected to hold in large generality, across different models and underlying graphs. (But they are proved mathematically in very few cases only.) We will briefly discuss one of them in Subsection 4.3.

4.2 Near-critical behavior for $q \in (1, 4]$ and self-organized monotone coupling

Let us consider the random-cluster model on \mathbb{Z}^2 with fixed cluster-weight $q \in (1, 4]$ (we drop it for several notations). In this subsection, our goal is to illustrate that there is a strong self-organized mechanism within this monotone Markov process which goes beyond the specific heat effect. To show this, we will take for granted the specific heat exponent (which describes the blow-up of the derivative of the edge-intensity at criticality for $q \geq 2$,

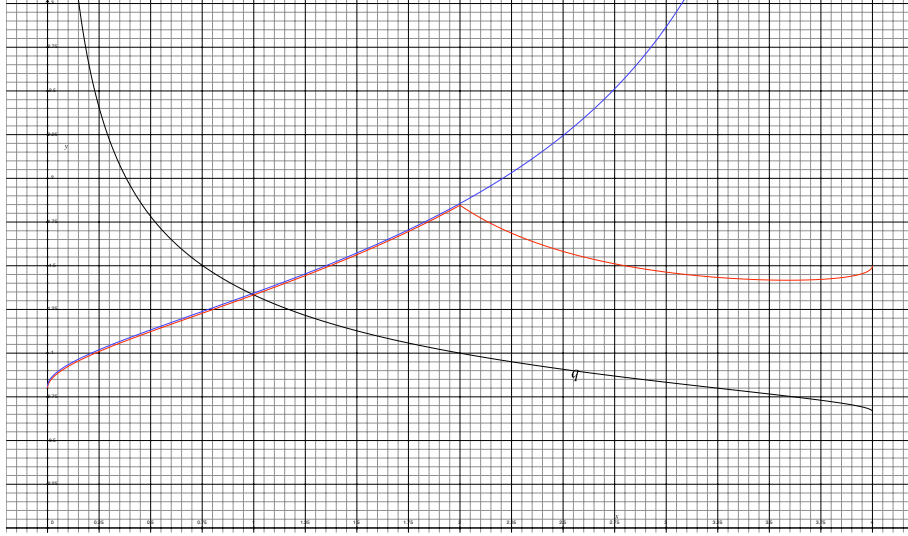


Figure 4.1: The blue curve corresponds to what the correlation length exponent in the critical random-cluster model with parameter q would be if the monotone coupling happened to be “Poissonian”. The red curve is a refinement of the blue one where the specific heat is taken into account. Finally, the black curve represents the actual correlation length exponent.

as mentioned in Subsection 3.2), and based on this, we will estimate what would be the correlation length exponent if there was no self-organized mechanism (i.e., if new edges simply arrived in a Poissonian way).

Let us first describe our setup: we will restrict ourselves to a finite but very large window $\Lambda_n := [-n, n]^2$ with, say, wired boundary conditions. Now, starting from a critical random-cluster configuration ω_{p_c} in Λ_n , we raise $p = p_c + \Delta p$ until macroscopic effects start being non-negligible. If p_0 is the value where we stop, we should thus obtain the relation $L(p_0) \asymp n$.

Now, the specific heat exponent $\alpha = \alpha(q)$ has the following interpretation when $q > 2$:

$$\frac{d}{dp} \mathcal{I}^{\text{FK}_q}(p) \asymp \left(\frac{1}{|p - p_c(q)|} \right)^{\alpha(q)},$$

where $\mathcal{I}^{\text{FK}_q}(p) := \phi_{\mathbb{Z}^2, p, q}(e \text{ is open})$ is the random-cluster edge-intensity. From this result (which is at the level of a prediction in the physics literature), it is reasonable to expect that in the finite volume range, one has

$$\frac{d}{dp} \mathcal{I}_n^{\text{FK}_q}(p) \asymp n^{\alpha(q)},$$

as far as $n \lesssim L(p)$ (where $\mathcal{I}_n^{\text{FK}_q}(p)$ denotes the edge-intensity in Λ_n with, say, wired boundary conditions). This is known in the case $q = 2$ (where $\alpha(2) = 0$ with logarithmic blow-ups), where the finite-size behavior matches with the infinite volume one, as we already mentioned in Subsection 3.2.

On the other hand, when $q \in [1, 2)$, the specific heat exponent $\alpha(q) \in [-2/3, 0)$ measures “only” the second order variation of the partition function of the random-cluster model

around $(p, q) = (p_c(q), q)$, and there is no blow-up of the derivative of the edge-intensity in this case:

$$\frac{d}{dp} \mathcal{I}_n^{\text{FK}_q}(p) \asymp 1,$$

as far as $n \lesssim L(p)$. In particular, the specific heat exponent in our analysis will play a role only for $q \in [2, 4]$. (See, however, the next subsection.)

Let us then start from a critical configuration ω_{p_c} in Λ_n (with wired conditions) and let us raise p to the level $p = p_c + \Delta p$ in such a way that one still has $n \lesssim L(p)$. From the above discussion, one expects that about $n^2 \Delta p n^{\alpha(q) \wedge 1}$ new edges will arrive. If we assume the absence of self-organization, *i.e.*, if edges arrive more or less independently of the current configuration (except possibly a local rate which would depend on whether the endpoints of the edge are connected or not), then each of these arrivals should be macroscopic pivotal flips with probability about $n^{-\xi_4(q)}$ (we implicitly harnessed the fact that the pivotal exponent does not vary below the critical length). Therefore, at $n \approx L(p)$, we expect

$$n^2 \Delta p n^{\alpha(q) \wedge 1} n^{-\xi_4(q)} \approx 1.$$

Let $L^{\text{Poiss}}(p)$ denote the correlation length obtained via the above analysis. We find

$$L^{\text{Poiss}}(p) \approx \left(\frac{1}{|p - p_c|} \right)^{\frac{1}{2 - \xi_4(q) + \alpha(q) \wedge 1}},$$

which is represented as a function of q by the red curve in Figure 4.1 (the blue curve represents the result of the same analysis when specific heat blow-ups are not taken into account).

As one can see from Figure 4.1, the actual correlation length $L(p)$ is much smaller than $L^{\text{Poiss}}(p)$ when $q \in (1, 4]$, which reveals for all these random-cluster models non-trivial self-organized schemes as p increases near $p_c(q)$.

4.3 The hyperscaling relation between correlation length and specific heat

There is the following well-known **hyperscaling relation** between correlation length and specific heat:

$$2 - \alpha = \nu d, \tag{4.1}$$

which is expected to hold for all q , provided that the dimension d of the underlying lattice \mathbb{Z}^d is low enough.

We have been arguing that the mere quantity of new edges arriving does not explain the correlation length alone, but the self-organized structure in which they arrive also matters. Nevertheless, the hyperscaling relation tells us that the correlation length is in fact determined by the change in the edge intensity, in some other way. Although we have not managed to relate the mechanism for the hyperscaling to self-organization, let us give a brief heuristic derivation of (4.1) in the case of $q \geq 2$ (where it is linked to the change in the edge intensity), in a way easily accessible to mathematicians.

On the infinite lattice, the derivative of the edge intensity is $d\mathcal{I}(p)/dp \approx |p - p_c|^{-\alpha}$. Since a finite system starts looking similar to the infinite system at the scale of the correlation length $L(p)$, it is reasonable to expect (as we did also in the previous subsection) that the

edge intensity is changing in a similar way in a finite system of side-length $L(p)$. Hence, the number of new edges in this box, when getting from p_c to p , should be about

$$\mathbb{E}_{[L(p)]^d, p} |\omega| - \mathbb{E}_{[L(p)]^d, p_c} |\omega| = L(p)^d |p - p_c|^{1-\alpha} \approx |p - p_c|^{\nu d + 1 - \alpha},$$

coming from integrating the above rate of change. On the other hand, similarly to (3.8), we have $\frac{d}{dp} \mathbb{E}_{[L(p)]^d, p} |\omega| \asymp \text{Var}_{[L(p)]^d, p} |\omega|$, which means that the standard deviation of $|\omega|$ from its mean should be around $|p - p_c|^{(-\nu d - \alpha)/2}$. The near-critical window should be given by the near-critical bias just starting to overcome the existing fluctuations, which results in the equation

$$-\nu d + 1 - \alpha = \frac{-\nu d - \alpha}{2},$$

yielding the desired identity (4.1).

For the percolation case ($q = 1$), Section 9.2 of [Gri99] contains a similarly vague heuristics, while [BCKS99] studies those hyperscaling relations that do not involve the specific heat.

5 On the dynamical sensitivity of random-cluster models

We end this paper with a slightly tangential discussion on the heat-bath dynamics for the random-cluster models with $q > 1$. A rigorous treatment of the special case $q = 2$ will be presented in [GP].

The purpose of the first subsection is to highlight the following interesting phenomenon:

Phenomenon 5.1 (conjectural). *There exist “natural” critical two-dimensional systems (i.e., scale-invariant and so on) with the property that they have pivotal points at all scales (hence are expected to be noise sensitive), but for which there are no exceptional times of infinite clusters along the natural heat-bath dynamics (hence are not dynamically sensitive).*

The second subsection is a rigorous treatment of the case $q > 25.72$ (which should then also hold for all $q > 4$ but is yet conjectural).

5.1 Noise- versus dynamical sensitivity for $q \in [1, 4]$

The (conjectural) Phenomenon 5.1 is rather surprising, since, in every model examined in the literature so far, noise sensitivity and dynamical sensitivity (i.e., the existence of exceptional times with an infinite cluster) have been present or absent together. One can see this as another illustration of the fact that a good understanding of the structure of the set of pivotal points is by far not enough to answer questions on noise- or dynamical sensitivity. Recall that all the current proofs of noise- and dynamical sensitivity for critical percolation ($q = 1$) rely on the “largeness” of the Fourier spectrum of percolation, a very different object compared to the pivotal points (see [GPS10a]). The Fourier spectrum of random-cluster models should still be “large”, i.e., concentrated on “high frequencies” once projected on an orthogonal basis of eigenfunctions of the heat-bath dynamics. Nevertheless, it seems that, when q gets closer to 4, the Fourier spectrum is not large enough to overcome the typical critical behavior (the smallness of the one-arm probability), hence exceptional times will not exist. We now discuss the reasons for this phenomenon in greater detail.

As for the dynamics for the monotone coupling, an infinite-volume heat-bath dynamics on \mathbb{Z}^2 can be constructed for every (p, q) with $q \geq 1$. These dynamics are constructed

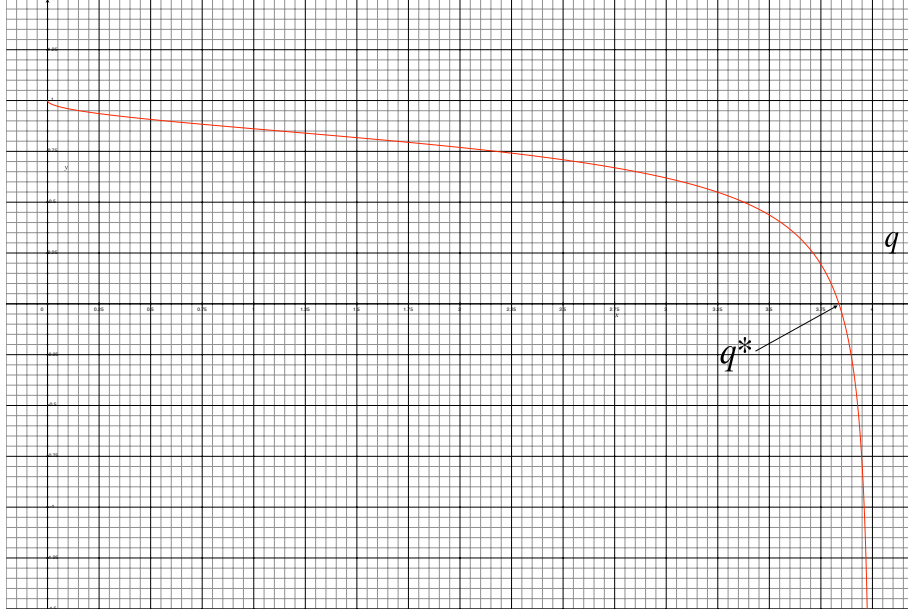


Figure 5.1: The red curve represents the upper bound on the Hausdorff dimension of exceptional times for random-cluster models, $q \in [1, 4]$ which is obtained heuristically in this subsection. It highlights an interesting transition as the variable q increases. Note that the red curve gives the correct bound of $31/36$ for $q = 1$ (proved in [GPS10a]).

as monotone limits of the *wired* or *free* finite volume heat-bath dynamics and are unique as soon as the infinite-volume measure is unique at (p, q) (even though, theoretically, they could differ from each other). Fix $q \in [1, 4]$ and $p = p_c(q)$. In this case, the infinite-volume measure is expected to be unique. Set $(\omega_t^{\text{FK}_q})_{t \geq 0}$ to be the unique critical dynamics obtained either from the free or the wired limit. The natural question raised by the discussion of the percolation case is the following:

Question 5. For $q \in (1, 4]$, are there exceptional times t almost surely for which there is an infinite cluster in $\omega_t^{\text{FK}_q}$? If “yes” and if \mathcal{E}_q denotes the random set of these exceptional times, what is the (almost sure) Hausdorff dimension of \mathcal{E}_q ?

We conjecture the following property for which we will then give a heuristic proof based on the above exponents from the physics literature.

Conjecture 5.2. Let

$$q^* := 4 \cos^2\left(\frac{\pi}{4}\sqrt{14}\right) \approx 3.83.$$

- For $q \in (q^*, 4]$, FK_q percolation is NOT dynamically sensitive.
- If $q \in [1, q^*]$, one has a.s.

$$\dim(\mathcal{E}_q) \leq \frac{1 - 8u(q) + 2u(q)^2}{3u(q)^2 - 8u(q)}, \quad (5.1)$$

where, as above, $u(q) := \frac{2}{\pi} \arccos(\frac{\sqrt{q}}{2}) = 2 - \frac{8}{\kappa(q)}$.

This conjecture hints that there is a critical $q_c \in [1, q^*]$, above which there are no exceptional times. It is natural to expect that $q \mapsto \mathbb{E}[\dim(\mathcal{E}_q)]$ is continuous and thus that $q_c > 1$. It is quite possible that $q_c = q^*$ and (5.1) is an equality.

Heuristic explanation of the conjecture. For standard percolation, the usual way to prove upper bounds on the dimension of exceptional times (and prove sometimes their non-existence) is to divide the time evolution into tiny intervals I of length ϵ each, dominate the union of critical configurations $\{\omega_t : t \in I\}$ by a slightly supercritical percolation configuration, and use the near-critical behavior. However, due to the complicated structure of the near-critical FK_q percolation highlighted in the previous section, one cannot easily do this for $q > 1$. Instead, one can rely on a more artificial near-critical version, where new edges are added to a critical random-cluster configuration in a Poissonian way. The correlation length $\tilde{L}(p)$ of this near-critical model can be derived from the *four-arm* critical exponent as in the case $q = 1$ (this assumes RSW and the analog of near-critical stability described in Phenomenon 1.5 and Subsection 3.1). This gives

$$\tilde{L}(p) \approx \left(\frac{1}{|p - p_c(q)|} \right)^{\frac{1}{2 - \xi_4(q)}}.$$

Below this correlation length, the artificial near-critical configuration has the same connectivity properties as a critical random-cluster configuration. In particular, if all our configurations $\omega_t^{\text{FK}_q}$, $t \in I$ are dominated by a configuration $\tilde{\omega}_{p_c + \epsilon}$, one obtains

$$\begin{aligned} \mathbb{P}[I \cap \mathcal{E}_q \neq \emptyset] &\leq \mathbb{P}[0 \xleftrightarrow{\omega_{p_c + \epsilon}} \tilde{L}(p_c + \epsilon)] \\ &\lesssim \mathbb{P}[0 \xleftrightarrow{\omega_{p_c}} \tilde{L}(p_c + \epsilon)] \\ &\approx \tilde{L}(p_c + \epsilon)^{-\xi_1(q)} \approx \epsilon^{\frac{\xi_1(q)}{2 - \xi_4(q)}}. \end{aligned}$$

Since one needs $O(\epsilon^{-1})$ intervals I to cover $[0, 1]$, a first moment argument implies that a.s.

$$\dim_H(\mathcal{E}_q) \leq 1 - \frac{\xi_1(q)}{2 - \xi_4(q)}.$$

Plugging in the expressions from Subsection 4.1 explains the second part of Conjecture 5.2. For the first part, it is enough to solve the equation $\frac{\xi_1(q)}{2 - \xi_4(q)} = 1$, so that above its solution, the dimension would be “negative”, i.e., we expect the set \mathcal{E}_q to be almost surely empty. The solution of this equation is given by q^* .

Finally, let us mention that (5.1) will be made rigorous in the special case $q = 2$ in the forthcoming paper [GP].

5.2 Random-cluster models with $q > 4$

The purpose of this subsection is to briefly explain what occurs when $q > 4$. In this case, the phase transition is expected to be of first order (proved for q large enough, *i.e.*, $q > 25.72$): in $\mathbb{P}_{p_c(q)}^{\text{FK}_{q,\text{free}}}$ there is no infinite cluster, while in $\mathbb{P}_{p_c(q)}^{\text{FK}_{q,\text{wired}}}$ there is. Another way to formulate this phenomenon is that, in a finite system $[-n, n]^d$, the critical parameter $p_c(q)$ of the infinite system is already outside the critical window: it is subcritical in the free case, supercritical in the wired case. However, the finite system is not more interesting even at the “true n -dependent critical value” (with any reasonable definition; say, the left-right crossing probability is exactly $1/2$): any infinite-volume measure should be a mixture of $\mathbb{P}_{p_c(q)}^{\text{FK}_{q,\text{free}}}$ and $\mathbb{P}_{p_c(q)}^{\text{FK}_{q,\text{wired}}}$ (again, proved in any dimension $d \geq 2$ for large enough q ,

see [LMMS⁺91] and the references there). In particular, the correlation length remains uniformly bounded for all $n \in \mathbb{N}$ and $p \in [0, 1]$. Therefore, most of the previous discussion of the near-critical regime does not make sense.

What one could still ask is the size of the critical window, and how it is related to the structure of the emerging clouds in the monotone coupling. It appears that the clouds are even larger than in the case $q \in (1, 4]$. First of all, Theorem 4.63 and Proposition 8.59 of [Gri06] say that there is an atom at p in the infinite volume monotone coupling if and only if there is non-uniqueness of the measure at that p . That is, in the infinite volume limit, a positive fraction of the edges appear at $p_c(q)$ for q large. (One can also prove this from the exponential decay of connectivity in the critical free measure, see (5.2) below.) A priori, for the finite systems this means only that there is an interval shrinking to p_c in which a positive fraction of the edges appear. However, the uniform boundedness of the correlation length strongly suggests that the transition from the subcritical to the supercritical phase can happen only by a simultaneous appearance of a positive density of edges; *i.e.*, there should be a uniformly positive atom even for the finite systems.

One can also discuss the possibility of noise and dynamical sensitivity. Yet, the exponential decay of correlations suggests that the model is not noise sensitive. As an illustration, we exploit the known exponential decay estimates for $\mathbb{P}_{p_c(q)}^{\text{FK}_q, \text{free}}$ for $q > 25.72$ in order to obtain the following result. Of course, the theorem is expected to hold for all $q > 4$.

Theorem 5.3. *When q is large enough, there are no exceptional times for the infinite free boundary heat-bath dynamics on critical FK_q configurations.*

Proof. The proof is based on [Gri06, Theorem 6.35] which states that in dimension $d = 2$, and if $q > 25.72$, then at the critical point $p_c(q) = p_{\text{sd}}(d)$, one has for the **free** infinite volume limit:

$$\phi_{p_c(q), q}^0(0 \longleftrightarrow \partial[-n, n]^2) \leq C \exp(-c(q)n), \quad (5.2)$$

where $c(q) > 0$ is a positive constant which depends only on $q > 25.72$. By monotonicity, this result implies that for the random-cluster measure on the finite box $\Lambda_N := [-N, N]^2$ endowed with free boundary conditions, then for all radius n such that $2n \leq N$, one has

$$\phi_{\Lambda_N, p_c(q), q}^0(x \longleftrightarrow \partial(x + \Lambda_n)) \leq C \exp(-c(q)n),$$

for all points $x \in \Lambda_n$. This in turn implies that for all $x \in \Lambda_n$, the probability to have a *four-arm* event around x of radius n is bounded above by the same exponential bound $\exp(-c(q)n)$.

Now, let us fix a large radius $n \gg 1$. Our goal is to find a small upper bound on

$$g(n) := \phi_{p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \xrightarrow{\omega_t} \partial\Lambda_n)$$

for the free boundary infinite-volume heat-bath dynamics on \mathbb{Z}^2 . Since, as we discussed earlier, this infinite volume limit is obtained as an increasing limit of finite volume heat-bath dynamics and since the event under consideration is a cylinder event, one has

$$g(n) = \lim_{N \rightarrow \infty} \phi_{\Lambda_N, p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \xrightarrow{\omega_t} \partial\Lambda_n).$$

Using the random variable $X_n = X_n^{(N)}$ to denote the number of flips over the time interval $[0, 1]$ for the event $\{0 \longleftrightarrow \partial\Lambda_n\}$, one easily obtains

$$\begin{aligned} \mathbb{E}[X_n] &\leq O(1) n^2 \sup_{x \in \Lambda_n} \phi_{\Lambda_N, p_c(q), q}^0(x \longleftrightarrow x + \partial(\Lambda_{n/2})) \\ &\leq O(1) n^2 \exp(-c(q)n/2). \end{aligned}$$

This in turn implies the bound

$$\phi_{\Lambda_N, p_c(q), q}^0(\exists t \in [0, 1] \text{ s.t. } 0 \xleftrightarrow{\omega_t} \partial \Lambda_n) \leq O(1) n^2 \exp(-c(q)n/2),$$

which gives a uniform upper bound in N . In particular, $g(n) \leq O(1) n^2 \exp(-c(q)n/2)$ and thus, taking $n \rightarrow \infty$, one concludes that a.s. there are no exceptional times for the critical free random-cluster measure on \mathbb{Z}^2 with q large enough. \square

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